The Nahm transform for calorons

Benoit Charbonneau and Jacques Hurtubise *

May 16, 2007.

Dedicated to Nigel Hitchin on the occasion of his sixtieth birthday.

Abstract

In this paper, we complete the proof of an equivalence given by Nye and Singer of the equivalence between calorons (instantons on $S^1 \times \mathbb{R}^3$) and solutions to Nahm's equations over the circle, both satisfying appropriate boundary conditions. Many of the key ingredients are provided by a third way of encoding the same data which involves twistors and complex geometry.

1 Introduction

One rather mysterious feature of the self-duality equations on \mathbb{R}^4 is the existence of a quite remarkable non-linear transform, the Nahm transform. It maps solutions to the self-duality equations on \mathbb{R}^4 invariant under a closed translation group $G \subset \mathbb{R}^4$ to solutions to the self-duality equations on $(\mathbb{R}^4)^*$ invariant under the dual group G^* . This transform uses spaces of solutions to the Dirac equation, it is quite sensitive to boundary conditions, which must be defined with care, and it is not straightforward: for example it tends to interchange rank and degree.

The transform was introduced by Nahm as an adaptation of the original ADHM construction of instantons ([3]) having the advantage that it can be generalised. The series of papers [26, 25, 10] details Nahm's original transform. In the case monopole case $G = \mathbb{R}$, the transform takes the monopole to a solution to some non-linear matrix valued o.d.e.s, Nahm's equations on an interval, and there is an inverse transform giving back the monopole. The mathematical development of this transform is due to Nigel Hitchin([16]), who showed for the SU(2) monopole how the monopole and the corresponding solution to Nahm's equations are both encoded quite remarkably in the same algebraic curve, the spectral curve of the monopole. This work was extended to the cases of monopoles for the other classical groups in [18]. This extension illustrates just how odd the transform can be: one gets solutions to the Nahm equations on a chain of intervals, but the size of the matrices jump from interval to interval.

The Nahm transform for other cases of G-invariance has been studied by various authors (among others [5, 6, 9, 20, 22, 21, 27, 29, 30]); a nice survey can be found in [23]. Here, we study the case of "minimal" invariance, under \mathbb{Z} ; the fields are referred to as calorons, and the Nahm transform sends them to solutions of Nahm's equations over the circle. This case is very close to the monopole one, and indeed calorons can be considered as Kač–Moody monopoles ([13]).

This project started while we were studying the work of Nye and Singer ([27, 28]) on calorons; they consider the Nahm transform directly, and do most of the work required to show that the transform is involutive. The missing ingredients turn out to lie in complex geometry, in precisely the same way Hitchin's extra ingredient of a spectral curve complements Nahm's. The complex geometry allows us to complete the equivalence, which can then be used to compute the moduli. It thus seems to us quite appropriate to consider this problem in a volume dedicated to Nigel Hitchin, as it allows us to revisit some of his beautiful mathematics, including some on calorons which has remained unpublished.

^{*}The first author is supported by an NSERC PDF. The second author is supported by NSERC and FQRNT grants. The diagrams in this paper were created using Paul Taylor's Commutative Diagram package.

In Section 2, we summarise the work of Nye and Singer towards showing that the Nahm transform is an equivalence between calorons and appropriate solutions to Nahm's equations. In Section 3, following in large part on work of Garland and Murray, we describe the complex geometry ("spectral data") that encodes a caloron. In Section 4, we study the process by which spectral data also correspond to solutions to Nahm's equations. In Section 5, we close the circle, showing the two Nahm transforms are inverses. In Section 6, we give a description of moduli, expounded in [8].

2 The work of Nye and Singer

2.1 Two types of invariant self-dual gauge fields on \mathbb{R}^4 .

Nye and Singer study the Nahm transform between the following two self-dual gauge fields (we restrict ourselves to SU(2), thought they study the more general case of SU(N)):

A- SU(2) Calorons of charge (k, j).

SU(2)-calorons are self-dual SU(2)-connections on $S^1 \times \mathbb{R}^3$, satisfying appropriate boundary conditions. We view $S^1 \times \mathbb{R}^3$ as the quotient of the standard Euclidean \mathbb{R}^4 by the time translation $(t, x) \mapsto (t + 2\pi/\mu_0, x)$. Let A be such a connection, defined over a rank 2 vector bundle V equipped with a unitary structure; we write it in coordinates over \mathbb{R}^4 as

$$A = \phi dt + \sum_{i=1,2,3} A_i dx_i,$$

with associated covariant derivatives

$$\nabla = \left(\frac{\partial}{\partial t} + \phi\right)dt + \sum_{i=1,2,3} \left(\frac{\partial}{\partial x_i} + A_i\right)dx_i \equiv \nabla_t dt + \sum_{i=1,2,3} \nabla_i dx_i.$$

We require that the L^2 norm of the curvature of A be finite, and that in suitable gauges, the A_i be $O(|x|^{-2})$, and that ϕ be conjugate to $diag(i(\mu_1 - j/2|x|), i(-\mu_1 + j/2|x|)) + O(|x|^{-2})$ for a positive real constant μ_1 and a positive integer j (the monopole charge). We also have bounds on the derivatives of these fields.

The boundary conditions tell us in essence that in a suitable way the connection extends to the 2-sphere at infinity times S^1 ; furthermore, one can show that the extension is to a fixed connection, which involves fixing a trivialisation at infinity; there is thus a second invariant we can define, the relative second Chern class, which we represent by a (positive) integer k. There are then two integer charges for our caloron, k and j, respectively the instanton and monopole charges.

There is a suitable group of gauge transformations acting on these fields [27]: Nye's approach is to compactify to $S^1 \times \bar{B}^3$ with a fixed trivilisation over the boundary $S^1 \times S^2$. The gauge transformations are those extending smoothly the identity on the boundary. Two solutions are considered to be equivalent if they are in the same orbit under this group.

B- Solutions to Nahm's equations on the circle.

The second class of objects we consider are skew adjoint matrix valued functions $T_i(z)$, i = 0, ..., 3, of size $(k + j) \times (k + j)$ over the interval $(-\mu_1, \mu_1)$, and of size $(k) \times (k)$ over the interval $(\mu_1, \mu_0 - \mu_1)$, (hence $\mu_1 < \mu_0 - \mu_1$, so we impose that condition), that are solutions to Nahm's equations

$$\frac{dT_{\sigma(1)}}{dz} + [T_0, T_{\sigma(1)}] = [T_{\sigma(2)}, T_{\sigma(3)}], \quad \text{for } \sigma \text{ even permutations of (123)},$$

on the circle $\mathbb{R}/(z\mapsto z+\mu_0)$. These equations are reductions of the self-duality equations to one dimension, and are invariant under a group of gauge transformations under which the $\frac{d}{dz}+T_0$ transforms as a connection. We think of the T_i as sections of the endomorphisms of a vector bundle K whose rank jumps at the two boundary points; at these points $\pm \mu_1$ we need boundary conditions.

Case 1: $j \neq 0$. At each of the boundary points, there is a large side, with a rank (k+j) bundle, and a small side, with a rank k bundle. We attach the two at the boundary point using an injection ι from the small side into the large side, and a surjection π going the other way, with $\pi \cdot \iota$ the identity. One asks, from the small side, that the T_i have well defined limits at the boundary point. From the large side, one has a decomposition near the boundary points of the bundle $\mathbb{C}^{k+j} \times [-\mu_1, \mu_1]$ into an orthogonal sum of subbundles of rank k and j invariant with respect to the connection $\frac{d}{dz} + T_0$ and compatible with the maps ι, π . With respect to this decomposition the T_i have the form

$$T_i = \begin{pmatrix} \hat{T}_i & O(\hat{z}^{(j-1)/2}) \\ O(\hat{z}^{(j-1)/2}) & R_i \end{pmatrix}, \tag{2}$$

in a basis where T_0 is gauged to zero and for a choice of coordinate $\hat{z} := z - (\pm \mu_1)$. At the boundary point $\hat{z} = 0$, each \hat{T}_i has a well defined limit that coincides using π, ι with the limit from the small side. The R_i have simple poles, and we ask that their residues form an irreducible representation of su(2).

Case 2: j = 0. Here the boundary conditions are different. We have at the boundary, identification of the fibres of K from both sides. With this identification, we ask that, in a gauge where T_0 is zero, the T_i have well defined limits $(T_i)_{\pm}$ from both sides, and that, setting

$$A(\zeta) = (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2,$$
(3)

one has that taking the limits from both sides,

$$A(\zeta)_{+} - A(\zeta)_{-} = (\alpha_0 + \alpha_1 \zeta)(\bar{\alpha}_1 - \bar{\alpha}_0 \zeta)^T \tag{4}$$

for vectors α_0, α_1 in \mathbb{C}^k . In particular, $A(\zeta)_+ - A(\zeta)_-$ is of rank at most one for all ζ .

In both cases, there is a symmetry condition, that, in a suitable gauge,

$$T_i(0) = T_i(0)^T$$

and, finally, an irreducibility condition, that there be no covariant constant sections of the bundle left invariant under the matrices T_i . Solutions for which there are such covariant constant sections should correspond to cases where instanton charge has bubbled off, leaving behind a caloron of lower charge.

2.2 The Nahm transform

From caloron to a solution to Nahm's equations.

Let z be a real parameter. Using the Pauli matrices e_i , we define the Dirac operators D_z acting on sections of the tensor product of the vector bundle V with the spin bundles $S_{\pm} \simeq \mathbb{C}^2 \times S^1 \times \mathbb{R}^3$ by

$$D_z \colon \Gamma(V \otimes S_+) \to \Gamma(V \otimes S_-)$$
$$s \mapsto (\nabla_t + iz)s + \sum_i (e_i \nabla_i)s.$$

The Weitzenbock formula $D_z D_z^* = \nabla^* \nabla + \rho(F_{-z}^{-})$ guarantees that D_z^* is injective. Nye and Singer show in [28] that for z not in the set of lifts $\pm \mu_1 + n\mu_0$ ($n \in \mathbb{Z}$) of the boundary point $\pm \mu_1$ from the circle to \mathbb{R} , and with suitable choices of function spaces, the operator D_z is Fredholm, and its index away from these points is k+j for z lying in the intervals $(-\mu_1 + n\mu_0, \mu_1 + n\mu_0)$ and is k in the intervals $(\mu_1 + n\mu_0, -\mu_1 + (n+1)\mu_0)$. There is thus a bundle over \mathbb{R} whose rank jumps at $\pm \mu_1 + n\mu_0$ ($n \in \mathbb{Z}$), with fibre $\ker(D_z)$ at z. There is a natural way of shifting by μ_0 , identifying $\ker(D_z)$ with $\ker(D_{z+\mu_0})$, giving a bundle K over the circle.

Over each interval, this bundle sits inside the trivial bundle whose fibre is the space of L^2 sections of $V \otimes S_-$. Let P be the orthogonal projection from this trivial bundle onto K. As elements of K decay exponentially, the operation X_i of multiplying by the coordinate x_i can be used to define operators on sections of K by

$$\frac{d}{dz} + T_0 = P \cdot \frac{d}{dz},$$
$$T_i = P \cdot X_i.$$

Theorem 2.1 ([27, Sec 4.1.2, p.108]) The operators defined in this way satisfy Nahm's equations.

This theorem and Theorem 2.2 below fall in line with the general Nahm transform heuristic philosophy: the curvature of the transformed object, seen as an invariant connection on \mathbb{R}^4 , is always composed of a self-dual piece and another piece depending on the behaviour at infinity of the harmonic spinors. Since these are decaying exponentially here, that other piece is zero and the transformed object is self-dual, or equivalently once we reduce, it satisfies Nahm's equations; see for instance [7, Sec. 3].

From a solution to Nahm's equations to a caloron.

For the inverse transform, we proceed the same way: from a solution to Nahm's equations, we define a family of auxiliary operators parameterised by points of \mathbb{R}^4 acting on sections of $K \otimes \mathbb{C}^2$ by

$$D_{x,t} = i\left(\frac{d}{dz} + T_0 - it + \sum_{j=1,2,3} e_j(T_j - ix_j)\right).$$

One must again distinguish two cases, j > 0 and j = 0.

j > 0: We define the space W of L_1^2 sections of $K \otimes \mathbb{C}^2$ such that at $\pm \mu_1$ the values of the sections coincide: for the sections s_1 on the small side and s_2 on the large side, we need $\iota(s_1) = s_2$. Let X be the space of L^2 sections of $K \otimes \mathbb{C}^2$ over the circle. Set

$$\hat{D}_{x,t} := D_{x,t} \colon W \to X \tag{5}$$

j=0: We have as above the space W of sections. We define in addition a two dimensional space U associated with the end points $\pm \mu_1$. The jump condition given by Equation (4) at μ_1 imply that there is a vector $u_+ \in V_{\mu_1} \otimes \mathbb{C}^2$ such that, as elements of $End(V_{\mu_1} \times \mathbb{C}^2)$,

$$\sum_{j=1}^{3} ((T_j)_+ - (T_j)_-) \otimes e_j = (u_+ \otimes u_+^*)_0.$$

Here the subscript ()₀ signifies taking the trace free $End(V) \otimes Sl(2,\mathbb{C})$ component inside $End(V) \otimes End(\mathbb{C}^2) \simeq End(V \otimes \mathbb{C}^2)$.

One has a similar vector u_- at $-\mu_1$. Let U be the vector space spanned by u_+, u_- . Let $\Pi: V_{\mu_1} \oplus V_{-\mu_1} \to U$ be the orthogonal projection, let X be the sum of the space of L^2 sections of V with the space U, and set

$$\hat{D}_{x,t} := (D_{x,t}, \Pi) \colon W \to X \tag{6}$$

The kernel of this operator consists of sections s in the kernel of $D_{x,t}$, with values at $\pm \mu_1$ lying in U^{\perp} ; the cokernel consists of triples $(s, c_{\mu_1} u_{\mu_1}, c_{-\mu_1} u_{-\mu_1})$, where s is a section of $K \otimes \mathbb{C}^2$, lying in the kernel of $D_{x,t}^*$, with jump discontinuity c_+u_+ at μ_1 , and c_-u_- at $-\mu_1$.

Theorem 2.2 ([27, p. 104]) The operator $\hat{D}_{x,t}$ has trivial kernel for all (x,t), and has a cokernel of rank 2, defining a rank 2 vector bundle over \mathbb{R}^4 , with natural time periodicity which allows one to build a vector bundle V over $S^1 \times \mathbb{R}^3$, defined locally as a subbundle of the infinite dimensional bundle $X \times S^1 \times \mathbb{R}^3$. Let P denote the orthogonal projection from X to V. Setting, on sections of V,

$$\nabla_i = P \cdot \frac{\partial}{\partial x_i}, \quad i = 1, 2, 3$$
$$\nabla_t = P \cdot \frac{\partial}{\partial t},$$

defines an SU(2)-caloron.

2.3 Involutivity of the transforms.

We would like these two transforms to be inverses of each other, so that they define an equivalence. The results of Nye and Singer get us most of the way there. What is missing is a proof that the solutions to Nahm's equations one obtains from a caloron satisfy the correct boundary and irreducibility conditions and then, that the two constructions are inverses to one another.

The easiest way to do so, as Hitchin did in his original paper on SU(2) monopoles, is to exploit the regularity given by a third equivalent set of data, which involves complex geometry. To do this, we examine some work of Garland and Murray, building on some unpublished work of Hitchin.

3 The twistor transform for calorons/Kač–Moody monopoles

3.1 Upstairs: the twistor transform for calorons.

Like all self-dual gauge fields on \mathbb{R}^4 or its quotients \mathbb{R}^4/G , calorons admit a twistor transform, translating the gauge fields into holomorphic vector bundles on an auxiliary space, the twistor space associated to \mathbb{R}^4/G . Following [13, Sec. 2], we summarise the construction, again for SU(2) only.

It is convenient first to recall from [15, Sec. 3] the twistor space for \mathbb{R}^3 . It can be interpreted as the space of oriented lines in \mathbb{R}^3 and it is isomorphic to $T\mathbb{P}^1$, the tangent bundle of the Riemann sphere. Let ζ be the standard affine coordinate on \mathbb{P}^1 , and η be the fibre coordinate in $T\mathbb{P}^1$ associated to the basis vector $\partial/\partial\zeta$. The incidence relation between the standard coordinates on \mathbb{R}^3 and (ζ, η) is given by

$$\eta = (ix_1 - x_2) + 2x_3\zeta + (ix_1 + x_2)\zeta^2.$$

The space $T\mathbb{P}^1$ comes equipped with standard line bundles $\mathcal{O}(k)$, lifted from \mathbb{P}^1 , and line bundles L^t , parameterised by $t \in \mathbb{C}$, with transition function $exp(t\eta/\zeta)$ from the open set $\zeta \neq \infty$ to the open set $\zeta \neq 0$. For $t \in \mathbb{R}$, these line bundles correspond to the standard U(1) monopoles on \mathbb{R}^3 given by a flat connection and constant Higgs field it. Let $L^t(k) := L^t \otimes \mathcal{O}(k)$. These bundles are in some sense the building blocks for monopoles for higher gauge groups.

According to Garland and Murray, the twistor space \mathcal{T} for $S^1 \times \mathbb{R}^3$ parametrises pairs consisting of a point in $S^1 \times \mathbb{R}^3$ and a unit vector in \mathbb{R}^3 . Such a pair gives a cylinder along which to integrate, and that projects to an oriented line in \mathbb{R}^3 . There is then a \mathbb{C}^* fibration $\pi \colon \mathcal{T} \to T\mathbb{P}^1$, which is in fact holomorphic and the complement of the zero section in L^{μ_0} . It has a natural fibrewise compactification $\tilde{\mathcal{T}} = \mathbb{P}(\mathcal{O} \oplus L^{\mu_0})$ compactifying the cylinders into spheres. The complement $\tilde{\mathcal{T}} \setminus \mathcal{T}$ is a sum of two disjoint divisors \mathcal{T}^0 and \mathcal{T}^∞ mapping isomorphically to $T\mathbb{P}^1$.

In the monopole case, the holomorphic vector bundle was obtained by integrating $\nabla_s - i\phi$ along real lines using a metric coordinate s along the line, and the Higgs field ϕ . Here, the analogous operation is integrating $\nabla_s - i\nabla_t$ over the cylinders to obtain a vector bundle E over \mathcal{T} . The boundary conditions allow us to extend the bundle to the compactification $\tilde{\mathcal{T}}$; we denote this extension again by E. The boundary conditions also give line subbundles over these divisors, $E_1^0 = L^{-\mu_1}(-j)$ over \mathcal{T}^0 , corresponding to solutions of $\nabla_s - i\phi$ which decay as one goes to plus infinity on the cylinder, and $E_1^\infty = L^{\mu_1}(-j)$ over \mathcal{T}^∞ , corresponding to solutions which decay as one goes to minus infinity.

The twistor space has a real structure τ , which acts on the bundle by $\tau^*(E) = E^*$, and maps the line subbundle $L^{-\mu_1}(-j)$ over \mathcal{T}^0 to the annihilator of the subbundle $L^{\mu_1}(-j)$ over \mathcal{T}^{∞} .

3.2 Downstairs: the caloron as a Kač-Moody monopole.

As Garland and Murray show in [13, Sec. 6], there is very nice way of thinking of the caloron as a monopole over \mathbb{R}^3 , with values in a Kač-Moody algebra, extending a loop algebra: the fourth variable t becomes the internal loop algebra variable. This way of thinking goes over to the twistor space picture.

Indeed, the twistor transform for the caloron gives us a bundle E over \mathcal{T} . Taking a direct image (restricting to sections with poles of finite order along $0, \infty$) gives one an infinite dimensional vector bundle F over

 $T\mathbb{P}^1$. If w is a standard fibre coordinate on \mathcal{T} vanishing over \mathcal{T}^0 and with a simple pole at \mathcal{T}^{∞} , it induces an endomorphism W of F, and quotienting F by the \mathcal{O} -module generated by the image of $W-w_0$ gives us the restriction of E to the section $w=w_0$, so that E and (F,W) are equivalent. One has more, however; the fact that the bundle E extends to $\tilde{\mathcal{T}}$ gives a subbundle F^0 of sections in the direct image which extend over \mathcal{T}^0 , and a subbundle F^{∞} of sections in the direct image which extend over \mathcal{T}^{∞} . One can go further and use the flags $0=E_0^0\subset E_1^0\subset E_2^0=E$ over \mathcal{T}^0 , $0=E_0^{\infty}\subset E_1^{\infty}\subset E_2^{\infty}=E$ over \mathcal{T}^{∞} to define for $p\in\mathbb{Z}$ and q=0,1 subbundles $F_{p,q}^0,F_{p,q}^{\infty}$ of F as

$$\begin{split} F^0_{-p,q} &= \{ s \in F \mid w^{-p}s \text{ finite at } \mathcal{T}^0 \text{ with value in } E^0_q \}, \\ F^\infty_{p,q} &= \{ s \in F \mid w^{-p}s \text{ finite at } \mathcal{T}^\infty \text{ with value in } E^\infty_q \}. \end{split}$$

We now have infinite flags

$$\cdots \subset F_{-1,0}^0 \subset F_{-1,1}^0 \subset F_{0,0}^0 \subset F_{0,1}^0 \subset F_{1,0}^0 \subset F_{1,1}^0 \subset \cdots
\cdots \supset F_{2,0}^\infty \supset F_{1,1}^\infty \supset F_{1,0}^\infty \supset F_{0,1}^\infty \supset F_{0,0}^\infty \supset F_{-1,1}^\infty \supset \cdots$$
(7)

Note that $W\cdot F^0_{p,q}=F^0_{p+1,q}$ and $W\cdot F^\infty_{p,q}=F^\infty_{p-1,q}.$ Garland and Murray show:

- $F_{p,0}^0$ and $F_{-p+1,0}^\infty$ have zero intersection and sum to F away from a compact curve S_0 lying in the linear system $|\mathcal{O}(2k)|$;
- $F_{p,1}^0$ and $F_{-p,1}^\infty$ have zero intersection and sum to F away from a compact curve S_1 lying in the linear system $|\mathcal{O}(2k+2j)|$;
- the quotients $F_{p,0}^0/F_{p-1,1}^0$ are line bundles isomorphic to $L^{(p-1)\mu_0+\mu_1}(j)$;
- the quotients $F_{v,1}^0/F_{v,0}^0$ are line bundles isomorphic to $L^{p\mu_0-\mu_1}(-j)$;
- the quotients $F_{p,0}^{\infty}/F_{p-1,1}^{\infty}$ are line bundles isomorphic to $L^{-(p-1)\mu_0-\mu_1}(j)$;
- the quotients $F_{p,1}^{\infty}/F_{p,0}^{\infty}$ are line bundles isomorphic to $L^{-p\mu_0+\mu_1}(-j)$.

It is worthwhile stepping back now and seeing what we have from a group theoretic point of view. On $T\mathbb{P}^1$, we have a bundle with structure group $G = \widetilde{Gl}(2,\mathbb{C})$ of $Gl(2,\mathbb{C})$ -valued loops. The flags that we have found give two reductions R_0, R_∞ to the opposite Borel subgroups B_0, B_∞ (of loops extending to $0, \infty$ respectively in \mathbb{P}^1 , and preserving flags over these points) in G. As for finite dimensional groups, we have exact sequences relating the Borel subgroups, their unipotent subgroups, and the maximal torus:

In a suitable basis, B_0 , B_{∞} consist respectively of upper and lower triangular matrices, and T of diagonal matrices

The two reductions are generically transverse, and fail to be transverse over the spectral curves of the caloron. This failure of transversality gives us geometrical data which encodes the bundle and hence the caloron. This approach is expounded in [19] for monopoles, but can be extended to calorons in a straightforward fashion. We summarise the construction here. The vector bundle F can be thought of as defining an element f of the cohomology set $H^1(T\mathbb{P}^1, G)$, and so a principal bundle P_f ; one simply thinks of f as the transition functions for P_f , in terms of Čech cohomology. The reductions to B_0 , B_∞ , can also be thought of as elements f_0 , f_∞ of $H^1(T\mathbb{P}^1, B_0)$, $H^1(T\mathbb{P}^1, B_\infty)$, respectively, or as principal bundles P_{f_0} , P_{f_∞} , or, since they are reductions of P_f , as sections R_0 of $P_f \times_G G/B_0 = P_{f_\infty} \times_{B_\infty} G/B_0$ and R_∞ of $P_0 \times_G G/B_\infty = P_{f_0} \times_{B_0} G/B_\infty$.

We note also that the elements f_0 , f_{∞} when projected to $H^1(T\mathbb{P}^1,T)$, give fixed elements α_0 , α_{∞} , in the sense that they are independent of the caloron, or rather depend only on the charges and the asymptotics, which we presume fixed. It is a consequence of the fact that the successive quotients in Sequence (7) depend only on the μ_i and the charges. Let A_0 , A_{∞} denote the principal T bundles associated to α_0 , α_{∞} .

Let $A_0(U_0)$ be the sheaf of sections of the U_0 bundle associated to A_0 by the action of T on U_0 . The cohomology set $H^1(T\mathbb{P}^1, A_0(U_0))$ describes ([19, Sec. 3]) the set of B_0 bundles mapping to α_0 . To find the bundle f_0 , and so f, then, one simply needs to have the appropriate class in $H^1(T\mathbb{P}^1, A_0(U_0))$.

Note that U_0 also serves as the big cell in the homogeneous space G/B_{∞} . Following [14], we think of G/B_{∞} as adding an infinity to U_0 , so if U_0 denotes the sheaf of holomorphic maps into U_0 , the sheaf \mathcal{M} of holomorphic maps into G/B_{∞} can be thought of as meromorphic maps into U_0 . Hence there is an sequence of sheaves of pointed sets (base points are chosen compatibly in both U_0 and G/B_{∞})

$$0 \to \mathcal{U}_0 \to \mathcal{M} \to \mathcal{P}r \to 0$$

defining the sheaf $\mathcal{P}r$ of principal parts. As B_0, T act on these sheaves, we can build the twisted versions

$$P_{f_0}(\mathcal{U}_0) \longrightarrow P_{f_0}(\mathcal{M}) \stackrel{\phi}{\longrightarrow} P_{f_0}(\mathcal{P}r),$$

 $A_0(\mathcal{U}_0) \longrightarrow A_0(\mathcal{M}) \longrightarrow A_0(\mathcal{P}r).$

As we are quotienting out the action of U_0 , we have $A_0(\mathcal{P}r) \simeq P_{f_0}(\mathcal{P}r)$.

We are now ready to define the *principal part data* of the caloron. The principal part data of the caloron is the image under ϕ of the class of the reduction given as a section R_{∞} of $P_{f_0} \times_{B_0} G/B_{\infty} = P_{f_0}(\mathcal{M})$:

$$\phi(R_{\infty}) \in H^0(A_0(\mathcal{P}r)).$$

To get back the caloron bundle from the principal part data, one takes the coboundary $\delta(\phi(R_{\infty}))$, in the obvious Čech sense, for the second sequence, obtaining a class in $H^1(T\mathbb{P}^1, A_0(\mathcal{U}_0))$. One checks that this is precisely the class corresponding to the bundle P_0 ; while this seems a bit surprising, the construction is fairly tautological, and is done in detail in [19, Sec. 3]. In our infinite dimensional context, the bundles are quite special, as they, and their reductions, are invariant under the shift operator W.

Of course, this construction does not tell us what the mysterious class $\phi(R_{\infty})$ actually corresponds to, but it turns out that, over a generic set of calorons, it is quite tractable. Indeed, the principal part data, as a sheaf, is supported over the spectral curves, and, if the curves have no common components and no multiple components, then the principal part data amounts to the following spectral data ([13]):

- The two spectral curves, S_0 and S_1 ;
- The ideal $\mathcal{I}_{S_0 \cap S_1}$, decomposing as $\mathcal{I}_{S_{01}} \cdot \mathcal{I}_{S_{10}}$; the real structure interchanges the two factors;
- An isomorphism of line bundles $\mathcal{O}[-S_{10}] \otimes L^{\mu_0-2\mu_1} \simeq \mathcal{O}[-S_{01}]$ over S_0 ;
- An isomorphism of line bundles $\mathcal{O}[-S_{01}] \otimes L^{2\mu_1} \simeq \mathcal{O}[-S_{10}]$ over S_1 ;
- A real structure on the line bundle $\mathcal{O}(2k+j-1)[-S_{01}]\otimes L^{\mu_1}|S_1$, lifting the real involution τ on $T\mathbb{P}^1$.

The structure of this data can be understood in terms of the Schubert structure of G/B_0 . The sheaf of principal parts, by definition, lives in the complement of U_0 , where the two flags cease to be transverse. In the case that concerns us here (remember the invariance under W), there are essentially two codimension one varieties at infinity to U_0 that we consider, whose pull-backs give the two spectral curves. The decomposition of $S_0 \cap S_1$ into two pieces is simply a pull-back of the Schubert structure from G/B_0 . For the line bundles, the basic idea is that in codimension one, the principal parts can be understood using embeddings of \mathbb{P}^1 into G/B_0 given by principal $Sl(2,\mathbb{C})$ s in G (see for instance [4, Sec. 4]), reducing the question at least locally to understanding principal parts for maps into \mathbb{P}^1 , a classical subject. For simple poles, the principal part of a map into \mathbb{P}^1 is encoded by its residue. Here, the poles are over the spectral curves, and globally, the principal part is then encoded as a section of a line bundle over each curve; we identify these below. A way of seeing that the spectral data splits into components localised over each curve is that we can choose appropriately two parabolic subgroups Par_0 , Par_1 and project from G/B_0 to G/Par_0 and G/Par_1 .

To see what the "residues" correspond to here, we exploit a description of the bundle first given in [18, Prop 1.12], the short exact sequence

The quotients at the right are supported over the spectral curves $(F/(F_{p,0}^0 + F_{-(p-1),0}^{\infty}))$ over S_0 , and $F/(F_{p,1}^0 + F_{-p,1}^{\infty})$ over S_1) and those in the middle are generically line bundles.

The shift operator W moves the quotients in the middle and last columns two steps down. The quotients are in fact direct image sheaves $R^1\tilde{\pi}_*$ of sheaves on $\tilde{\mathcal{T}}$ derived from E; for example, let $E_{p,0,-(p-1),\infty}$ be the sheaf of sections of E over \mathcal{T} with poles of order p at \mathcal{T}_0 and a zero of order p-1 at \mathcal{T}_∞ . The sheaf $F/(F_{p,0}^0+F_{-p,1}^\infty)$ can be identified with $R^1\tilde{\pi}_*(E_{p,0,-(p-1),\infty})$. One can make similar identifications for the other sheaves. In a way that parallels [16], we have the following result.

Proposition 3.1 a) The spectral curves S_0 and S_1 are real $(\tau$ -invariant); the quotient $F/(F_{p,1}^0 + F_{-p,1}^\infty)$ over S_1 inherits from E a quaternionic structure, lifting τ .

b) The sheaves $F/(F_{p,0}^0 + F_{-(p-1),0}^{\infty})$ and $F/(F_{p,1}^0 + F_{-p,1}^{\infty})$ satisfy a vanishing theorem:

$$H^{0}\left(T\mathbb{P}^{1}, F/(F_{p,0}^{0} + F_{-(p-1),0}^{\infty}) \otimes L^{-z}(-2)\right) = 0, \text{ for } z \in [(p-1)\mu_{0} + \mu_{1}, p\mu_{0} - \mu_{1}],$$

$$H^{0}\left(T\mathbb{P}^{1}, F/(F_{p,1}^{0} + F_{-p,1}^{\infty}) \otimes L^{-z}(-2)\right) = 0, \text{ for } z \in (p\mu_{0} - \mu_{1}, p\mu_{0} + \mu_{1}).$$

The vanishing hold because these cohomology groups encode L^2 solutions to the Laplace equation in the caloron background, which must be zero; see [16, Thm 3.7] and [18, Thm 1.17].

For generic spectral curves, following a line of argument of [18], Garland and Murray show directly that the spectral data determines the caloron. To this end, they identify in [13, Sec. 6] the quotients

$$F/(F_{p,0}^{0} + F_{-(p-1),0}^{\infty}) = L^{p\mu_{0}-\mu_{1}}(2k+j)[-S_{10}]|_{S_{0}} = L^{(p-1)\mu_{0}+\mu_{1}}(2k+j)[-S_{01}]|_{S_{0}},$$

$$F/(F_{p,1}^{0} + F_{-p,1}^{\infty}) = L^{p\mu_{0}+\mu_{1}}(2k+j)[-S_{01}]|_{S_{1}} = L^{p\mu_{0}-\mu_{1}}(2k+j)[-S_{10}]|_{S_{1}},$$

$$F/(F_{p,0}^{0} + F_{-p,1}^{\infty}) = L^{p\mu_{0}-\mu_{1}}(2k+j) \otimes \mathcal{I}_{S_{10}},$$

$$F/(F_{p,1}^{0} + F_{-p,0}^{\infty}) = L^{p\mu_{0}+\mu_{1}}(2k+j) \otimes \mathcal{I}_{S_{01}}.$$

These identifications realised in exact sequence (9) give

The "residues" are then the sheaves on the right, supported over the spectral curves; the maps from the middle to the right hand sheaves give the various isomorphisms. This diagram shows that F, and hence the caloron, is encoded in the spectral data.

More generally, we can define the spectral data composed of the curves S_0, S_1 , the pull-backs to $T\mathbb{P}^1$ of the Schubert varieties in G/B_0 , and the sheaves on the right hand side of Sequence (9), with isomorphisms similar to the ones for generic spectral data given by maps from the sheaves in the middle column.

To summarise, we have that the caloron determines principal part data, a section of a sheaf of principal parts supported over the spectral curve; this data determines the caloron in turn. In the generic case, the principal part data is equivalent to spectral data can be described in terms of two curves, their intersections, and sections of line bundles over these curves. We note that these generic calorons exist; indeed, we already know that a caloron is determined by a solution to Nahm's equations, and it is easy to see that generic solutions to Nahm's equations exist, as we shall see in Section 6.

4 From Nahm's equations to spectral data, and back

4.1 Flows of sheaves.

The solutions to Nahm's equations we consider also determine equivalent spectral data, as we shall see. To begin, note that by setting $A(\zeta, z)$ as in Equation (3), and

$$A_{+}(\zeta, z) = -iT_{3}(z) + (T_{1} - iT_{2})(z)\zeta,$$

Nahm's equations (1) are equivalent to the Lax form

$$\frac{dA}{dz} + [A_+, A] = 0. {(11)}$$

The evolution of A is by conjugation, so the spectral curve given by

$$det(A(\zeta, z) - \eta \mathbf{I}) = 0 \tag{12}$$

in $T\mathbb{P}^1$ is an obvious invariant of the flow we have; if the matrices are $k \times k$, the curve is a k-fold branched cover of \mathbb{P}^1 . We define for each z a sheaf \mathcal{L}_z over the curve via the exact sequence

$$0 \to \mathcal{O}(-2)^k \xrightarrow{A(\zeta, z) - \eta \mathbf{I}} \mathcal{O}^k \to \mathcal{L}_z \to 0. \tag{13}$$

This correspondence taking a solution to Nahm's equations to a curve and a flow of line bundles over the curve is fundamental to the theory of Lax equations; see [1]. In our case, we have an equivalence:

Proposition 4.1 There is an equivalence between

- A) Solutions to Nahm's equations on an interval (a,b), given by $k \times k$ matrices with reality condition built from skew-hermitian matrices T_i as in Equation (3).
- B) Spectral curves S that are compact and lie in the linear system $|\mathcal{O}(2k)|$ lying in $T\mathbb{P}^1$, and flows \mathcal{L}_z , for $z \in (a,b)$, of sheaves supported on S, such that
 - $H^0(T\mathbb{P}^1, \mathcal{L}_z(-1)) = H^1(T\mathbb{P}^1, \mathcal{L}_z(-1)) = 0;$
 - $\mathcal{L}_z = \mathcal{L}_{z'} \otimes L^{z-z'}$, for $z, z' \in (a, b)$;
 - the curve S is real, that is, invariant under the antiholomorphic involution $\tau(\eta,\zeta) = (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$ corresponding to reversal of lines in $T\mathbb{P}^1$:
 - there is a linear form μ on $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z))$ vanishing on $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C))$ for all fibres C of $T\mathbb{P}^1 \to \mathbb{P}^1$ and inducing a positive definite hermitian metric $(\sigma_1, \sigma_2) \mapsto \mu(\sigma_1 \tau^*(\sigma_2))$ on $H^0(S \cap C, \mathcal{L}_z)$.

We start with the passage from A) to B). For $H^0(T\mathbb{P}^1, \mathcal{L}_z(-1)) = H^1(T\mathbb{P}^1, \mathcal{L}_z(-1)) = 0$ to hold, we need $\mathcal{O}(-3)^k \xrightarrow{A(\zeta,z)-\eta \mathbf{I}} \mathcal{O}(-1)^k$ to induce an isomorphism on H^1 . As shown in [18, Lemma 1.2], the groups $H^1(T\mathbb{P}^1, \mathcal{O}(p))$ are infinite dimensional spaces nicely filtered by finite dimensional pieces, corresponding

to powers of η in the cocycle. A basis for $H^1(T\mathbb{P}^1, \mathcal{O}(-3))$ is $1/\zeta, 1/\zeta^2, \eta/\zeta, \ldots, \eta/\zeta^4, \eta^2/\zeta, \ldots, \eta^2/\zeta^6, \ldots$, and a basis for $H^1(T\mathbb{P}^1, \mathcal{O}(-1))$ is obtained from it by multiplying by η . Multiplication by η then induces an isomorphism, and so multiplication by $A(\zeta, z) - \eta \mathbf{I}$ gives an isomorphism $H^1(T\mathbb{P}^1, \mathcal{O}(-3)^k) \to H^1(T\mathbb{P}^1, \mathcal{O}(-1)^k)$. For the relation $\mathcal{L}_z = \mathcal{L}_{z'} \otimes L^{z-z'}$, see [18, Sec. 2.6] or [1] for the general theory of the Lax flows. The reality of the curve follows from the fact that the T_i are skew hermitian. For the final property, the positive definite inner product on \mathcal{O}^k with respect to which the T_i are skew hermitian induces one on $H^0(T\mathbb{P}^1, \mathcal{L}_z)$, by passing to global sections in Sequence (13). The inner product is a linear map $H^0(S, \mathcal{L}_z) \otimes H^0(S, \tau^*(\mathcal{L}_z)) \to \mathbb{C}$. As \mathcal{L}_z represents, in essence, the dual to the eigenspaces of the $A(\zeta)^T$, this linear map factors through $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z))$, and it must be zero on sections that vanish on fibres. Now that the passage from A) to B) is established, note that since $\tau^*L^{z-z'} = L^{z'-z}$, the product $\mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)$ is constant. The last condition of B) imposes severe constraints on \mathcal{L}_z : when S is smooth, and \mathcal{L}_z a line bundle, it tells us that

$$\mathcal{L}_z \otimes \tau^*(\mathcal{L}_z) \simeq K_S(2C).$$
 (14)

Indeed, in that case the vanishing of the cohomology of $\mathcal{L}_z(-1)$ tells us $\deg(\mathcal{L}_z) = g + k - 1$ and $\deg(\mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)) = 2g + 2k - 2$. The space of sections $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z))$ is of dimension g + 2k - 1, while $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C))$ and $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C'))$ are of dimension g + k - 1, and $\dim H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C - C')) = g - 1$, unless it is the canonical bundle, in which case the dimension is g. There cannot be a non-zero form on $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z))$ vanishing on $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C)) + H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C'))$ unless the intersection $H^0(S, \mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C - C'))$ is of dimension g, in which case $\mathcal{L}_z \otimes \tau^*(\mathcal{L}_z)(-C - C')$ is the canonical bundle.

Now we deal with the passage B) to A). In essence, it has been treated by Hitchin in [16], but we give a more invariant construction of A suitable for our purposes. The fibre product FP of $T\mathbb{P}^1$ with itself over \mathbb{P}^1 is the vector bundle $\mathcal{O}(2) \oplus \mathcal{O}(2)$, with fibre coordinates η, η' , and projections π, π' to $T\mathbb{P}^1$; the lift of $\mathcal{O}(2)$ to this vector bundle has global sections $1, \zeta, \zeta^2, \eta, \eta'$, and the diagonal Δ is cut out by $\eta - \eta'$. Hence over FP we have the exact sequence

$$0 \to \mathcal{O}(-2) \xrightarrow{\eta - \eta'} \mathcal{O} \to \mathcal{O}_{\Delta} \to 0. \tag{15}$$

Lifting \mathcal{L}_z via π , tensoring it with this sequence, and pushing it down via π' gives

$$0 \to V(-2) \xrightarrow{\underline{A}(\zeta) - \eta'} V \to \mathcal{L}_z \to 0 \tag{16}$$

for some rank k vector bundle V, which over $(\zeta, \eta) = (\zeta_0, \eta_0)$ is the space of sections of \mathcal{L} in the fibre of $T\mathbb{P}^1$ over $\zeta = \zeta_0$, and so is independent of η : V is lifted from \mathbb{P}^1 . We then identify V. The fibre product $FP = \mathcal{O}(2) \oplus \mathcal{O}(2)$ lives in the product $T\mathbb{P}^1 \times T\mathbb{P}^1$, and is cut out by $\zeta - \zeta'$, and one has the exact sequence

$$0 \to \mathcal{O}(-1, -1) \xrightarrow{\zeta - \zeta'} \mathcal{O} \to \mathcal{O}_{FP} \to 0. \tag{17}$$

Lifting \mathcal{L}_z to $T\mathbb{P}^1 \times T\mathbb{P}^1$, tensoring it with this sequence and taking direct image on the other factor gives the long exact sequence

$$0 \to H^0(T\mathbb{P}^1, \mathcal{L}_z(-1)) \otimes \mathcal{O}(-1) \to H^0(T\mathbb{P}^1, \mathcal{L}_z) \otimes \mathcal{O} \to V \to H^1(T\mathbb{P}^1, \mathcal{L}_z(-1)) \otimes \mathcal{O}(-1) \cdots$$

The vanishing of the cohomolology of $\mathcal{L}_z(-1)$ shows that $V \simeq \mathcal{O}^k$.

We then have given $\underline{A}(\zeta, z)$ as a map of bundles; the process of turning it into a matrix valued function $A(\zeta, z)$, and of showing that it evolves according to Nahm's equations as one tensors it by L^t , is given in [16, Prop. 4.16]. Similarly, the definition of an inner product on $H^0(T\mathbb{P}^1, \mathcal{L}_z)$ and the proof that the T_i are skew adjoint with respect to the inner product follow the line given in [16, Sec. 6].

4.2 Boundary conditions.

Now suppose that we have a solution to Nahm's equations satisfying the boundary conditions for an SU(2) caloron, as given above. We shall show that the boundary behaviour allows us to identify the initial

conditions for the flow, in other words to say what the sheaves \mathcal{L}_z are. As there are two intervals, there are two spectral curves, S_0 , which is a k-fold cover of \mathbb{P}^1 , and S_1 , which is k+j-fold. We suppose the curves are generic, in that they have no common components and no multiple components. A solution to Nahm's equations is called generic if its curves are.

Proposition 4.2 There is an equivalence between

- A) Generic solutions $A(\zeta, z)$ to Nahm's equations over the circle, denoted $A^0(\zeta, z)$ on $(\mu_1, -\mu_1 + \mu_0)$ and $A^1(\zeta, z)$ on $(-\mu_1, \mu_1)$, satisfying the conditions of Section 2.1-B.
- B) Generic spectral curves S_0 in $T\mathbb{P}^1$ of degree k over \mathbb{P}^1 that are the support of sheaves \mathcal{L}_z^0 , $z \in [\mu_1, \mu_0 \mu_1]$, and spectral curves S_1 in $T\mathbb{P}^1$ of degree k+j over \mathbb{P}^1 that are the support of sheaves \mathcal{L}_z^1 , $z \in [-\mu_1, \mu_1]$, with the following properties:
 - \mathcal{L}_z^0 , for $z \in [\mu_1, \mu_0 \mu_1]$, \mathcal{L}_z^1 , for $z \in (-\mu_1, \mu_1)$, S_0 , and S_1 have the properties of Proposition 4.1.
 - The intersection $S_0 \cap S_1$ decomposes as a sum of two divisors S_{01} and S_{10} , interchanged by τ .
 - $At \mu_1 = \mu_0 \mu_1$ (on the circle), $\mathcal{L}^0_{-\mu_1} = \mathcal{O}(2k+j-1)[-S_{10}]|_{S_0}$ and $\mathcal{L}^1_{-\mu_1} = \mathcal{O}(2k+j-1)[-S_{10}]|_{S_1}$.
 - At μ_1 , $\mathcal{L}_{\mu_1}^0 = \mathcal{O}(2k+j-1)[-S_{01}]|_{S_0}$ and $\mathcal{L}_{\mu_1}^1 = \mathcal{O}(2k+j-1)[-S_{01}]|_{S_1}$.
 - There is a real structure on \mathcal{L}_0^1 lifting τ .

As a consequence there are isomorphisms of line bundles $\mathcal{O}[-S_{01}] \otimes L^{\mu_0-2\mu_1} \simeq \mathcal{O}[-S_{10}]$ over S_0 and $\mathcal{O}[-S_{10}] \otimes L^{2\mu_1} \simeq \mathcal{O}[-S_{01}]$ over S_1 .

Let us begin with a discussion of the case k = 0, studied by Hitchin in [16]. He showed that the solution to Nahm's equations for an SU(2) monopole corresponded to the flow of line bundles $L^{t+\mu_1}(j-1)$, $t \in [-\mu_1, \mu_1]$ on the monopole's spectral curve, with the flow being regular in the middle of the interval, and having simple poles at the ends of the interval, with residues giving an irreducible representation of SU(2).

In the construction given in Proposition 4.1, we obtain for this flow a bundle V over $\mathbb{P}^1 \times [-\mu_1, \mu_1]$, and a regular section of Hom(V(-2), V) over this interval. For $t \in (-\mu_1, \mu_1)$ the bundle V is trivial on $\mathbb{P}^1 \times \{t\}$. The singularity at the end of the interval is caused by a jump in the holomorphic structure in V at the ends. (Both ends are identical, because of the identification $\mathcal{O} = L^{2\mu_1}$ over the spectral curve.)

To understand the structure of V at $\mathcal{L}_{-\mu_1} = \mathcal{O}(j-1)$, we note that in the natural trivialisations lifted from \mathbb{P}^1 , local sections of $\mathcal{O}(j-1)$ over $T\mathbb{P}_1$ are filtered by the order of vanishing along the zero section:

$$\mathcal{O}(j-1) \supset \eta \mathcal{O}(j-3) \supset \eta^2 \mathcal{O}(j-5) \supset \cdots;$$

this filtration can be turned onto a sum, as we have the subsheaves L_i of sections $\eta^i s$, with s lifted from \mathbb{P}^1 . The construction, by limiting to a curve which is a j-sheeted cover of \mathbb{P}^1 , essentially says that the degrees of vanishing of interest are at most j-1, as one is taking the remainder by division by the equation of the curve. The bundle V over $\mathbb{P}^1 \times \{\mu_1\}$ decomposes as a sum

$$V = \mathcal{O}(i-1) \oplus \mathcal{O}(i-3) \oplus \mathcal{O}(i-5) \oplus \cdots \oplus \mathcal{O}(-i+1).$$

More generally, for later use, set

$$V_{k,k-2\ell} \stackrel{\text{def}}{=} \mathcal{O}(k) \oplus \mathcal{O}(k-2) \oplus \mathcal{O}(k-4) \oplus \cdots \oplus \mathcal{O}(k-2\ell),$$

so $V = V_{j-1,-j+1}$. Writing $0 = \eta^j + \eta^{j-1}p_1(\zeta) + \cdots + p_j(\zeta)$ the equation of the spectral curve, with polynomials p_i of degree 2i, we can write the induced automorphism $A(\zeta) - \eta \mathbf{I}$ in this decomposition as

$$\begin{pmatrix}
-\eta & 0 & 0 & \dots & 0 & -p_{j}(\zeta) \\
1 & -\eta & 0 & \dots & 0 & -p_{j-1}(\zeta) \\
0 & 1 & -\eta & \dots & 0 & -p_{j-2}(\zeta) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & -\eta - p_{1}(\zeta)
\end{pmatrix}$$

There is another way of understanding how the pole of Nahm's equations generates a non-trivial V. We first restrict to $\zeta = 0$, and vary z. For a moment, suppose by translation that z = 0 is the point where the structure jumps. From the boundary conditions, the residues at 0 are given in a suitable basis by

$$Res(A_{+}) = diag(\frac{-(j-1)}{2}, \frac{2-(j-1)}{2}, \dots, \frac{(j-1)}{2}),$$
 (18)

$$Res(A) = \begin{pmatrix} 0 & 0 & \ddots & 0 & 0 \\ 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{19}$$

As in [17, Prop 1.15], we solve $\frac{ds}{dz} + A_+ s = 0$ along $\zeta = 0$, for s of the form $z^{\frac{j-1}{2}}((1,0,\ldots,0) + z \cdot \text{holomorphic})$. The sections $A^j s$ also solve the equation, and, using these sections as a basis, one conjugates $A - \eta \mathbf{I}$ to

$$\begin{pmatrix}
-\eta & 0 & 0 & \dots & 0 & -p_{j}(0) \\
1 & -\eta & 0 & \dots & 0 & -p_{j-1}(0) \\
0 & 1 & -\eta & \dots & 0 & -p_{j-2}(0) \\
\vdots & \vdots & \ddots & & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1 & -\eta - p_{1}(0)
\end{pmatrix}$$
(20)

using a matrix of the form M(z) = (Holomorphic in z)N(z), with

$$N(z) := diag(z^{\frac{-(j-1)}{2}}, z^{\frac{2-(j-1)}{2}}, \dots, z^{\frac{(j-1)}{2}}).$$
(21)

This process can be applied over any point ζ , as the matrices $A(\zeta)$, $A_+(\zeta)$ have residues (in z) conjugate to (18), (19). Indeed, at $\zeta = 0$, they represent standard generators of a representation of Sl(2), and moving away from $\zeta = 0$ simply amounts to a change of basis. Explicitly, conjugating the residues of $A(\zeta)$, $A_+(\zeta)$ by $N(\zeta)$ takes them to the residues of $\zeta A(1)$, $A_+(1)$. Conjugating again by a matrix T takes them to the residues of $\zeta A(0)$, $A_+(0)$, and then by $N(\zeta)^{-1}$ to the residues of A(0), $A_+(0)$. Thus, for a suitable

$$M(\zeta, z) = (\text{Holomorphic in } z, \zeta) \cdot N(\zeta^{-1}z)TN(\zeta) \tag{22}$$

one has

$$M(\zeta, z)(A(\zeta, z) - \eta \mathbf{I})M(\zeta, z)^{-1} = \begin{pmatrix} -\eta & 0 & 0 & \dots & 0 & -p_{j}(\zeta) \\ 1 & -\eta & 0 & \dots & 0 & -p_{j-1}(\zeta) \\ 0 & 1 & -\eta & \dots & 0 & -p_{j-2}(\zeta) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\eta - p_{1}(\zeta) \end{pmatrix}.$$
(23)

At z = 0, it is a holomorphic section of Hom(V(-2), V) in standard trivialisations.

The same procedures work when $k \neq 0$. Indeed, the singular solution to $\frac{ds}{dz} + A_+ s$ has the same behaviour. Starting from a solution to Nahm's equations, and integrating the connection as in [17], we find that, near μ_1 on the interval $(-\mu_1, \mu_1)$, a change of basis of the form

$$M(\zeta, z) = \text{(holomorphic in } z, \zeta) \cdot diag(\mathbf{I}_{k \times k}, N(\zeta^{-1}z)TN(\zeta))$$
 (24)

conjugates $A^{1}(\zeta, z) - \eta \mathbf{I}$ to the constant (in z) matrix

$$\begin{pmatrix}
a_{11}(\zeta) - \eta & a_{12}(\zeta) & \dots & a_{1k}(\zeta) & 0 & 0 & 0 & \dots & 0 & g_{1}(\zeta) \\
a_{21}(\zeta) & a_{22}(\zeta) - \eta & \dots & a_{1k}(\zeta) & 0 & 0 & 0 & \dots & 0 & g_{2}(\zeta) \\
\vdots & \vdots \\
a_{k1}(\zeta) & a_{k2}(\zeta) & \dots & a_{kk}(\zeta) - \eta & 0 & 0 & 0 & \dots & 0 & g_{k}(\zeta) \\
f_{1}(\zeta) & f_{2}(\zeta) & \dots & f_{k}(\zeta) & -\eta & 0 & 0 & \dots & 0 & -p_{j}(\zeta) \\
0 & 0 & \dots & 0 & 1 & -\eta & 0 & \dots & 0 & -p_{j-1}(\zeta) \\
0 & 0 & \dots & 0 & 0 & 1 & -\eta & \dots & 0 & -p_{j-2}(\zeta) \\
\vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\eta - p_{1}(\zeta)
\end{pmatrix}.$$
(25)

Setting $C(p(\zeta))$ to denote the companion matrix of $p(\zeta)$, we write this matrix schematically as

$$A^{1}(\zeta) - \eta \mathbf{I} = \begin{pmatrix} A^{0}(\zeta) - \eta \mathbf{I} & (0 & G(\zeta)) \\ F(\zeta) & C(p(\zeta)) - \eta \mathbf{I} \end{pmatrix}.$$
 (26)

Let M_{adj} denote the classical adjoint of M, so that $M_{adj}M = \det(M)\mathbf{I}$. Then

$$\det(A^{1}(\zeta) - \eta \mathbf{I}) = \det(A^{0}(\zeta) - \eta \mathbf{I}) \left(\eta^{j} + \sum \eta^{j-i} p_{i}(\zeta) \right) + (-1)^{j} F \left(A^{0}(\zeta) - \eta \mathbf{I} \right)_{adj} G. \tag{27}$$

The matrix $A^0(\zeta)$ is equal to the limit $A^0(\zeta, \mu_1)$. At the boundary point μ_1 , the bundle $V_{\mu_1}^0$ for S_0 is trivial, since the solution on S_0 is smooth at that point; one has

$$0 \to \mathcal{O}(-2)^k \xrightarrow{A^0(\zeta) - \eta \mathbf{I}} \mathcal{O}^k \to \mathcal{L}^0_{\mu_1} \to 0.$$
 (28)

The limit bundle $V_{\mu_1}^1$ for S_1 is

$$V_{u_{1}}^{1} = \mathcal{O}^{k} \oplus V_{j-1,-j+1} \tag{29}$$

with

$$0 \to V_{\mu_1}^1(-2) \xrightarrow{A^1(\zeta) - \eta \mathbf{I}} V_{\mu_1}^1 \to \mathcal{L}_{\mu_1}^1 \to 0.$$
 (30)

We now want to identify the limit bundles $\mathcal{L}_{\mu_1}^0$, $\mathcal{L}_{\mu_1}^1$, and the gluing between them. There is a divisor D contained in the intersection $S_0 \cap S_1$ such that $\mathcal{L}_{\mu_1}^0 \simeq \mathcal{O}(2k+j-1)[-D]|_{S_0}$ and $\mathcal{L}_{\mu_1}^1 \simeq \mathcal{O}(2k+j-1)[-D]|_{S_1}$, and the correspondence between the matrices is mediated by the maps

$$\mathcal{O}(2k+j-1)[-D]|_{S_0} \leftarrow \mathcal{O}(2k+j-1) \otimes \mathcal{I}_D \rightarrow \mathcal{O}(2k+j-1)[-D]|_{S_1}.$$

To prove its existence, we need to understand how to lift and push down $\mathcal{O}(2k+j-1)\otimes\mathcal{I}_D$ (here \mathcal{I}_D is the sheaf of ideals of D on $T\mathbb{P}^1$) and so $\mathcal{O}(2k+j-1)$ through Sequence (15). To do so, we compactify $T\mathbb{P}^1$ by embedding it into $\mathbf{T} = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$, adding a divisor at infinity P_{∞} . Let C be the fibre, then $P_{\infty} + 2C$ is linearly equivalent to the zero section P_0 of $T\mathbb{P}^1$. Our fibre product now compactifies to a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over \mathbb{P}^1 . Similarly, Sequence (15) compactifies to

$$0 \to \mathcal{O}(-2C - P_{\infty} - P_{\infty}') \xrightarrow{\eta - \eta'} \mathcal{O} \to \mathcal{O}_{\Delta} \to 0. \tag{31}$$

The line bundle $\mathcal{O}(mC + nP_{\infty})$ over **T** has over each fibre C a (n+1)-dimensional space of sections, and these sections are graded by the order in η , as before. Define an $\ell \times (\ell-1)$ matrix $S(\ell,\eta)$ by

$$S(\ell, \eta) = \begin{pmatrix} -\eta & 0 & \cdots & \cdots & 0 \\ 1 & -\eta & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\eta & 0 \\ \vdots & & \ddots & 1 & -\eta \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$
 (32)

Lemma 4.3 Let m, n > 0. Lifting $\mathcal{O}(mC + nP_{\infty})$ to the fibre product and tensoring with Sequence (28), then pushing down, we obtain an exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{n-1} \mathcal{O}((m-2i-2)C + (n-i-1)P_{\infty}) \xrightarrow{S(n,\eta)} \bigoplus_{i=0}^{n} \mathcal{O}((m-2i)C + (n-i)P_{\infty})$$
$$\xrightarrow{(1,\eta,\eta^2,\ldots,\eta^n)} \mathcal{O}(mC+nP_{\infty}) \to 0.$$

Over $T\mathbb{P}^1$, it becomes

$$0 \to V_{m-2,m-2n} \xrightarrow{S(n,\eta)} V_{m,m-2n} \xrightarrow{(1,\eta,\eta^2,\ldots,\eta^n)} \mathcal{O}(m) \to 0.$$

The proof is straightforward. We now want to define a subsheaf of $\mathcal{O}(2k+j)$. Set

$$R(\zeta, \eta) = \begin{pmatrix} A^{0}(\zeta) - \eta \mathbf{I} & 0 \\ \begin{pmatrix} F \\ 0 \end{pmatrix} & S(j, \eta) \end{pmatrix}$$

We define a vector of polynomial functions in (ζ, η)

$$(\phi_1, \dots, \phi_k) = -(-1)^k F(\zeta) (A^0(\zeta) - \eta \mathbf{I})_{adj}.$$

We have

$$(\phi_1, \dots, \phi_k) \left(A^0(\zeta) - \eta \mathbf{I} \right) + F(\zeta) (-1)^k \det \left(A^0(\zeta) - \eta \mathbf{I} \right) = 0.$$
(33)

Write $\phi_i = \sum_{j=0}^{k-1} \phi_{ji}(\zeta) \eta^j$, and $(-1)^k \det(A^0(\zeta) - \eta \mathbf{I}) = \eta^k + \sum_{j=0}^{k-1} h_j(\zeta) \eta^j$. Decomposing Equation (33) into different powers of η , we obtain

$$-\begin{pmatrix} 0 & \cdots & 0 \\ \phi_{01} & \cdots & \phi_{0k} \\ \vdots & & \vdots \\ \phi_{k-1,1} & \cdots & \phi_{k-1,k} \end{pmatrix} + \begin{pmatrix} \phi_{01} & \cdots & \phi_{0k} \\ \vdots & & \vdots \\ \phi_{k-1,1} & \cdots & \phi_{k-1,k} \\ 0 & \cdots & 0 \end{pmatrix} A^{0}(\zeta) + \begin{pmatrix} h_{0} \\ \vdots \\ h_{k-1} \\ 1 \end{pmatrix} F = 0.$$
 (34)

Let M_n be the $(k+n) \times (k+n)$ matrix

$$M_{n} = \begin{pmatrix} \phi_{01} & \dots & \phi_{0k} & h_{0} & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & 0 \\ \phi_{k-1,1} & \dots & \phi_{k-1,k} & h_{k-1} & & h_{0} \\ 0 & \dots & 0 & 1 & \ddots & \vdots \\ \vdots & & \vdots & 0 & \ddots & h_{k-1} \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}.$$
(35)

Using Equation (34), we have the commuting diagram

$$\mathcal{O}(-2)^{\oplus k} \oplus V_{j-3,-j+1} \xrightarrow{R(\zeta,\eta)} \mathcal{O}^{\oplus k} \oplus V_{j-1,-j+1} \xrightarrow{\mathcal{R}} \mathcal{R}$$

$$M_{j-1} \downarrow \qquad \qquad M_{j} \downarrow \qquad \qquad \downarrow$$

$$V_{2k+j-3,-j+1} \xrightarrow{S(k+j,\eta)} V_{2k+j-1,-j+1} \xrightarrow{(1,\eta,\ldots,\eta^{k+j-1})} \mathcal{O}(2k+j-1).$$

It defines a rank one sheaf \mathcal{R} , and embeds it in $\mathcal{O}(2k+j-1)$. This embedding fails to be surjective when $(1, \eta, \dots, \eta^{k+j-1})M_j = 0$, or equivalently when

$$F(A^0(\zeta) - \eta \mathbf{I})_{adj} = 0$$
 and $\det(A^0(\zeta) - \eta \mathbf{I}) = 0$.

These conditions can only be met on S_0 , and, because of Equation (27), on S_1 . In short, there is a subvariety D of $S_0 \cap S_1$, with $\mathcal{R} = \mathcal{O}(2k+j-1) \otimes \mathcal{I}_D$. There are natural surjective maps from \mathcal{R} to $\mathcal{L}^0_{\mu_1}$, $\mathcal{L}^1_{\mu_1}$: for the projection $\Pi_{m+n,n}$ on a sum with m+n summands onto the first n summands and the injection $I_{n,n+m}$ into the first n summands, we have that the diagram of exact sequences

$$\mathcal{O}(-2)^{\oplus k} \xrightarrow{A^{0}(\zeta) - \eta \mathbf{I}} \mathcal{O}^{\oplus k} \xrightarrow{\mathcal{L}_{\mu_{1}}^{0}} \mathcal{L}_{\mu_{1}}^{0}$$

$$\Pi_{k+j-1,k} \uparrow \qquad \Pi_{k+j,k} \uparrow \qquad \uparrow$$

$$\mathcal{O}(-2)^{\oplus k} \oplus V_{j-3,-j+1} \xrightarrow{R(\zeta,\eta)} \mathcal{O}^{\oplus k} \oplus V_{j-1,-j+1} \xrightarrow{\mathcal{R}} \mathcal{R}$$

$$I_{k+j-1,k+j} \downarrow \qquad I \downarrow \qquad \downarrow$$

$$\mathcal{O}(-2)^{\oplus k} \oplus V_{j-3,-j-1} \xrightarrow{A^{1}(\zeta) - \eta \mathbf{I}} \mathcal{O}^{\oplus k} \oplus V_{j-1,-j+1} \xrightarrow{\mathcal{L}_{\mu_{1}}^{1}} \mathcal{L}_{\mu_{1}}^{1}.$$

commutes. This diagram identifies $\mathcal{L}^0_{\mu_1}$, $\mathcal{L}^1_{\mu_1}$ as $\mathcal{O}(2k+j-1)[-D]$ over their respective curves.

A similar analysis for $-\mu_1$ shows that $\mathcal{L}^0_{-\mu_1}$, $\mathcal{L}^1_{-\mu_1}$ are $\mathcal{O}(2k+j-1)[-D']$, for some D'. To relate D to D', note that as in [16, Sec. 6], there is a real structure on \mathcal{L}_0 lifting the real involution τ . Hence $T_i(0) = T_i(0)^T$. On the other hand, for a solution A(z) to Nahm's equations (11),

$$\frac{dA(-z)^T}{dz} + [A_+(-z)^T, A(-z)^T] = 0.$$
(36)

As solutions to the same differential equation with the same initial condition, all the way around the circle,

$$A(-z) = A^{T}(z). (37)$$

This symmetry tells us that in terms of the matrix $A^1(\zeta)$, $A^0(\zeta)$ at μ_1 , the equation for D' is

$$0 = (A^0(\zeta) - \eta \mathbf{I})_{adi}G. \tag{38}$$

Using Equation (27), we see that along S_0 where $\det(A^0(\zeta) - \eta \mathbf{I}) = 0$, the intersection $S_0 \cap S_1$ is cut out by $0 = F(A^0(\zeta) - \eta \mathbf{I})_{adj}G$. Generically along S_0 , the matrix $(A^0(\zeta) - \eta \mathbf{I})$ has corank one, and so $(A^0(\zeta) - \eta \mathbf{I})_{adj}$ has rank one. We can thus write it as the product UV of a column vector and a row vector. The equation for the intersection is then 0 = (FU)(VG), the product of the defining relations for D and D'. If the two curves are smooth, intersecting transversally, then $S_0 \cap S_1 = D + D'$. This property holds independently of whether the matrices arise from calorons or not. As one has the result for generic intersections, varying continuously gives $S_0 \cap S_1 = D + D'$ even for non-generic curves.

The skew Hermitian property of the T_i implies that $A(\zeta, z) = -\zeta^2 \bar{A}(-1/\bar{\zeta}, z)^T$. This symmetry transforms the equation $0 = F(A^0(-1/\bar{\zeta}) - \eta \mathbf{I})_{adj}$ into $0 = (A^0(\zeta) - \eta \mathbf{I})_{adj}G$, showing that $\tau(D) = D'$.

We have now obtained our complete generic spectral data from our generic solution to Nahm's equations. The converse, starting with the spectral data, is essentially done above, apart from the boundary behaviour. This last piece is dealt with in [18, Sec. 2]; the conditions on the intersections of the spectral curves used here are more general, but the proof goes through unchanged.

5 Closing the circle

We have two different types of gauge fields, with a transform relating them; both have complex data associated to them, which encodes them; this data, we have seen (at least in the generic case) satisfies the same conditions, with for example the sheaves $(F/(F_{p,0}^0 + F_{-(p-1),0}^{\infty})) \otimes L^z(-1)$ and $(F/(F_{p,1}^0 + F_{-p,1}^{\infty})) \otimes L^z(-1)$ obtained from the caloron satisfying the same conditions as the \mathcal{L}^t obtained from the solution to Nahm's equation for appropriate values of z and t. We now want to check that the complex data associated to the object is the same as that associated to the object's transform.

5.1 Starting with a caloron.

Starting with a caloron, one can define as above, the spectral curves and sheaves over them.

- The curve S_0 is the locus where $F_{p,0}^0$ and $F_{-p+1,0}^{\infty}$ have non-zero intersection.
- The curve S_1 is the locus where $F_{p,1}^0$ and $F_{-p,1}^\infty$ have non-zero intersection.
- The quotient $F/(F_{p,0}^0 + F_{-(p-1),0}^{\infty})$, supported over S_0 , is generically isomorphic to the line bundle $L^{p\mu_0-\mu_1}(2k+j)[-S_{10}]|_{S_0} = L^{(p-1)\mu_0+\mu_1}(2k+j)[-S_{01}]|_{S_0}$.
- The quotient $F/(F_{p,1}^0 + F_{-p,1}^{\infty})$, supported over S_1 , is generically isomorphic to the line bundle $L^{p\mu_0+\mu_1}(2k+j)[-S_{01}]|_{S_1} = L^{p\mu_0-\mu_1}(2k+j)[-S_{10}]|_{S_1}$.

We can "shift" the caloron by the U(1) monopole with constant Higgs field iz. Let us consider the direction in \mathbb{R}^3 , corresponding to $\zeta=0$: the positive x_3 direction. If one does this, for $z\in ((p-1)\mu_0+\mu_1,p\mu_0-\mu_1)$, $F^0_{p,0}$ represents the sections in the kernel of $\nabla_3-i\nabla_0+z$ along each cylinder that decay at infinity in the positive direction; similarly, $F^\infty_{-(p-1),0}$ represents the sections along each cylinder that decay at infinity in the negative direction. For $z\in (p\mu_0-\mu_1,p\mu_0+\mu_1)$, $F^0_{p,1}$ represents the sections along each cylinder that decay at infinity in the positive direction; similarly, $F^\infty_{-p,1}$ represents the sections along each cylinder that decay at infinity in the negative direction. With these shifts, the spectral curves represent the lines for which the solutions that decay at one or other of the ends do not sum to the whole bundle. The quotients $F/(F^0_{p,0}+F^\infty_{-(p-1),0})$ and $F/(F^0_{p,1}+F^\infty_{-p,1})$ represent this failure and therefore the existence of a solution decaying at both ends. Suppose that $z\in (p\mu_0-\mu_1,p\mu_0+\mu_1)$; the other case is identical.

Following the Nahm transform heuristic, starting with the caloron, we solve the Dirac equation for the family of connections shifted by z, and obtain in doing so a bundle over an interval and a solution to Nahm's equations on this bundle. The spectral curve, from this point of view, over $\zeta = 0$, consist of the eigenvalues of $A(0,z) = T_1 + iT_2(z)$; there is a sheaf over the curve whose fibre over the point η in the spectral curve given by the cokernel of $A(0,z) - \eta \mathbf{I}$.

A priori, it is not evident what link there is between eigenvalues on a space of solutions to an equation defined over all of space, and the behaviour of solutions to an equation along a single line. The link is provided by a remarkable formulation of the Dirac equation; see for instance [12] or [9]. The idea, roughly, is to write the Dirac equation as the equations for the harmonic elements of the complex

$$L^{2}(V) \xrightarrow{D_{1} = \begin{pmatrix} \nabla_{3} + i\nabla_{0} + z \\ -\nabla_{1} - i\nabla_{2} \end{pmatrix}} L^{2}(V)^{\oplus 2} \xrightarrow{D_{2} = \begin{pmatrix} \nabla_{1} + i\nabla_{2} & \nabla_{3} + i\nabla_{0} + z \end{pmatrix}} L^{2}(V). \tag{39}$$

Proposition 5.1 The operators D_1, D_2 commute with multiplication by $w = x_1 + ix_2$, and for self-dual connections, $D_2D_1 = 0$. The kernel $K_z = \ker(D_2) \cap \ker(D_1^*)$ of the Dirac operator D_z^* is naturally isomorphic to the cohomology $\ker(D_2)/\operatorname{im}(D_1)$ of the complex.

On a compact manifold, this equivalence is part of standard elliptic theory. In the non-compact case, the analysis must be done with care. The simplest way, for us, is to adapt the work of Nye and Singer [28, Sec. 4]. Using the techniques developed by Mazzeo and Melrose [24], they show that the Dirac operator D_z^* is Fredholm if and only an "operator on fibers along the boundary" $P = (\nabla_0 - z)_{\infty} + i(\eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3)$ is invertible along all the circles S^1 in $S^1 \times S^2_{\infty} = \partial(S^1 \times \mathbb{R}^3)$. Again, the e_1 are the Pauli matrices, and the constraint must hold for all choices of constants (η_1, η_2, η_3) . Nye and Singer show that it does as long as z does not have values $\mu_i + n\mu_0$. In the the elliptic complex, the constraint gets replaced by the essentially equivalent condition that the complex

$$L^{2}(V|_{S^{1}}) \xrightarrow{\begin{pmatrix} \eta_{3} + i(\nabla_{0})_{\infty} + z \\ -\eta_{1} - i\eta_{2} \end{pmatrix}} L^{2}(V|_{S^{1}})^{\oplus 2} \xrightarrow{(\eta_{1} + i\eta_{2} \quad \eta_{3} + i(\nabla_{0})_{\infty} + z)} L^{2}(V|_{S^{1}})$$

defined over the circle be exact. The equivalence between the Euler characteristic of the complex and the index of the Dirac operator then goes through, establishing the isomorphism.

We now turn to analysing the way a solution to Nahm's equations is extracted from this complex. The operator A(0,z) on the kernel is defined by multiplication by $x_1 + ix_2$ followed by the projection on that kernel. On the cohomology, it is simply multiplication by $w = x_1 + ix_2$, simplifying matters considerably.

Suppose η is an eigenvalue of A(0,z), hence there is a cohomology class v such that $(x_1+ix_2-\eta)(v)=D_1(s)$ for some s in L^2 . In particular, along $w=\eta$ in $S^1\times\mathbb{R}^3$, we have $(\nabla_3+i\nabla_0-z)s=0$, and so $(\eta,\zeta)=(w,0)$ belongs to the spectral curve. Conversely, if $(\nabla_3+i\nabla_0-z)s=0$ along $w=\eta$, we can build a cohomology class v by extending s to a neighbourhood so that it is compactly supported in the x_1,x_2 directions and satisfies $(\nabla_1+i\nabla_2)s=0$ along $w=\eta$; one then sets $v=\frac{D_1s}{w-\eta}$. The spectral curves are thus identified.

We can further identify sections of the quotients $F/(F_{p,1}^0 + F_{-p,1}^\infty)$ (for the caloron twisted by z) over $S_1 \cap \{\zeta = 0\}$ with sections of the cokernel of $A^1(0,z) - \eta \mathbf{I}$ as both as supported over $S_1 \cap \{\zeta = 0\}$. Indeed, let $\sigma_{\epsilon}(w)$ and $\rho(x_3)$ be smooth functions such that

$$\sigma(w) = \begin{cases} 1, & |w| < \epsilon; \\ 0, & |w| > \epsilon; \end{cases} \text{ and } \rho(x_3) = \begin{cases} 0, & x_3 < 0; \\ 1, & x_3 > 1. \end{cases}$$

Notice that the derivative $(\nabla_1 + i\nabla_2)(\sigma_{\epsilon}(w))$ is supported on the annulus $\epsilon < |w| < 2\epsilon$.

Consider a ball B centred at η in the fiber above $\zeta = 0$ of radius 2ϵ chosen such that $B \cap S_1 = \{\eta\}$. The ball parameterises cylinders $S^1 \times \{w\} \times \mathbb{R}$, for $w \in B$, and thus a section $\phi \in H^0(B, F)$ is in fact a section of V on $S^1 \times B \times \mathbb{R}$ satisfying $(\nabla_3 + i\nabla_0 + z)\phi = 0$. The holomorphicity condition is then $(\nabla_1 + i\nabla_2)\phi = 0$. Let $\phi_0 \in H^0(B, F_{p,1}^0|_{(0,\eta)})$. Then $\sigma(w - \eta)\rho(x_3)\phi_0$ lies in $L^2(S^1 \times \mathbb{R}^3, V)$. Thus the section $D_1(\sigma(w - \eta)\rho(x_3)\phi_0)$ is a coboundary for Complex (39). Similarly, if ϕ_∞ lies in $F_{-p,1}^\infty$, then $\sigma(w - \eta)(\rho(x_3) - 1)\phi_\infty$ also lies in L^2 , and its image by D_1 is also a coboundary.

Suppose for simplicity $S_1 \cap \{\zeta = 0\}$ contains only points of multiplicity one. Consider now a general section $\phi \in H^0(B, F)$. Away from the spectral curve, ϕ decomposes into a sum $\phi_0 + \phi_\infty$, and combining the two constructions above gives a coboundary $K(\phi)$ for Complex (39). This decomposition has a pole at η if $\phi(\eta)$ represents a non trivial element in the quotient $F/(F_{p,1}^0 + F_{-p,1}^\infty)$. However, the section

$$K(\phi) = D_1 \left(\sigma_{\epsilon}(w - \eta) \rho(x_3) \phi - \rho_{\infty} \right)$$

$$= \begin{pmatrix} \sigma_{\epsilon}(w - \eta) \left(\nabla_3 + i \nabla_0 + z \right) \left(\rho(x_3) \phi - \rho_{\infty} \right) \\ - \left(\nabla_1 + i \nabla_2 \right) \left(\sigma_{\epsilon}(w - \eta) \rho(x_3) \phi - \rho_{\infty} \right) \end{pmatrix}$$

is L^2 . Because of the pole of $\phi_0 + \phi_\infty$, it is not in $D_1(L^2(V))$ but by construction it is definitely in the kernel of D_2 . It thus represents a non-trivial cohomology class for Complex (39).

Doing this for each point of $S_1 \cap \{\zeta = 0\}$ expresses K_z as a sum of classes localised along the lines in $S^1 \times \mathbb{R}^3$ corresponding to the intersection of the spectral curve with $\zeta = 0$, identifying K_z with sections of $F/(F_{p,1}^0 + F_{-p,1}^\infty)$ over $\zeta = 0$. Let $K_z(\phi)$ correspond to a non-zero element ϕ of $F/(F_{p,1}^0 + F_{-p,1}^\infty)$ over $(\eta, 0)$. This class projects non-trivially to the cokernel of $A(0,z) - \eta \mathbf{I}$. Indeed, the action of A(0,z) on K_z is multiplication by $w = x_1 + ix_2$, so if $K_z(\phi) = (A(0,z) - \eta \mathbf{I})K_z(\sigma) = (w - \eta)K_z(\sigma)$ for some σ with $\phi = (w - \eta)\sigma$, then the decomposition $\phi = \phi_0 + \phi_1$ is holomorphic and so $K_z(\phi) = 0$.

For intersections of higher multiplicity, the case presents only notational difficulties, but is conceptually identical. The choice of ζ made above just simplified the analysis, varying it we obtain the following result.

Proposition 5.2 For $z \in (p\mu_0 - \mu_1, p\mu_0 + \mu_1)$, the curves S_1 associated to the caloron and S'_1 associated to its Nahm transform are identical; there is a natural isomorphism between the sheaf $F/(F^0_{p,1} + F^\infty_{-p,1})$ for the caloron shifted by z and the cokernel sheaf of $A^1(\zeta, z) - \eta$.

Similarly, for $z \in ((p-1)\mu_0 + \mu_1, p\mu_0 - \mu_1)$, the curves S_0 associated to the caloron and S_0' associated to its Nahm transform are identical; there is a natural isomorphism between the sheaf $F/(F_{p,0}^0 + F_{-(p-1),0}^{\infty})$ for the caloron shifted by z and the cokernel sheaf of $A^0(\zeta, z) - \eta$.

We remark, in particular, for generic solutions, that flowing to a boundary point gives the divisor D in the intersection of the two curves, and so an identification of the spectral data.

5.2 Starting with a solution to Nahm's equation.

We now want to identify the spectral data for a solution to Nahm's equations and the caloron it produces. **The case** j > 0. Again, as for the opposite direction, it is be more convenient to give a cohomological interpretation of the Dirac equation. This interpretation is governed by choosing a particular direction in \mathbb{R}^3 , and so a particular equivalence of $S^1 \times \mathbb{R}^3$ with $\mathbb{C}^* \times \mathbb{C}$. We choose the direction corresponding to $\zeta = 0$, so set $\beta = T_1 + iT_2$, $w = x_1 + ix_2$, $\alpha = T_0 - iT_3$, $y = \mu_0 x_0 - ix_3$, and consider the complex

$$\tilde{L}_{2}^{2} \xrightarrow{D_{1} = \begin{pmatrix} i(d_{z} + \alpha - iy) \\ -\beta + iw \end{pmatrix}} (\tilde{L}_{1}^{2})^{\oplus 2} \xrightarrow{D_{2} = (\beta - iw \quad i(d_{z} + \alpha - iy))} \tilde{L}^{2}. \tag{40}$$

For this complex to be defined, we must display a bit of care in our choice of function spaces:

- \tilde{L}_2^2 is the space of sections s of K that are L_2^2 over each interval and such that, for subscripts denoting the limits on the appropriates sides of the jump points, $\pi(s_{\text{large}}) = s_{\text{small}}$, $\pi(d_z s_{\text{large}}) = d_z s_{\text{small}}$;
- $(\tilde{L}_1^2)^{\oplus 2}$ is the space of sections (s_1, s_2) of $K \oplus K$ that are in L_1^2 over each interval, with $\pi(s_{\text{large}}) = s_{\text{small}}$ at the jump points;
- \tilde{L}^2 is the space of L^2 sections of K.

Proposition 5.3 For solutions to Nahm's equations, $D_2D_1 = 0$, and

$$D_1^* D_1 = -\left(d_z + (T_0 - i\mu_0 x_0)\right)^2 - \sum_{j=1}^3 (T_j - ix_j)^2.$$
(41)

So for $v \neq 0$ satisfying $D_1^*D_1(v) = 0$, there is a point-wise relation

$$d_z^2 \|v\|^2 = 2\| \left(d_z + (T_0 - i\mu_0 x_0) \right) v\|^2 + 2 \sum_{j=1}^3 \| (T_j - ix_j) v\|^2 \ge 0.$$
 (42)

Hence if v is a regular solution vanishing at two points, v = 0.

We look at solutions to the equations $D_1^*D_1(v)=0$ near a boundary point $\pm \mu_1$; one has on the interval $(-\mu_1, \mu_1)$ near $\pm \mu_1$, a decomposition $\mathbb{C}^{k+j}=\mathbb{C}^k\oplus\mathbb{C}^j$; the 2k dimensional space of solutions with boundary values (and derivatives) in \mathbb{C}^k continue outside the interval, and solutions with boundary values in \mathbb{C}^j are governed by the theory of regular singular points for o.d.e.s. As the sum of the squares of the residues at the boundary points of the T_i is half of the Casimir element in the enveloping algebra of su(2), its value is $-(j-1)^2/4$. The theory of regular singular o.d.e. then gives, for ρ one of $(1\pm j)/2$, j dimensional spaces of solutions of the form $\hat{z}^\rho f(\hat{z})$ to $D_1^*D_1(v)=0$, where $f(\hat{z})$ is analytic, at each of the boundary points. Here \hat{z} is a coordinate whose origin is at the boundary point.

We can use this knowledge to build a Green's function. This operation is somewhat complicated by the poles of the T_i , as we now see in a series of lemmas.

Lemma 5.4 Let $x \in (-\mu_1, \mu_1)$, and $u \in \mathbb{C}^{k+j}$. There is a unique solution to $D_1^*D_1(v) = \delta_x u$ on the circle.

Proof: Such a solution must be continuous, smooth on the circle with a single jump in the derivative at x, of value u. Let U be the 2k-dimensional space of solutions on the small side, outside $(-\mu_1, \mu_1)$. Those solutions propagate into the interval from both ends, and for $f \in U$, let f_- be the continuation into the interval $(-\mu_1, \mu_1)$ from the $-\mu_1$ side, and f_+ from the μ_1 side. Let V_+ and V_- be the j-dimensional spaces of solutions of the form $\hat{z}^{(1+j)/2}f(\hat{z})$ respectively born at μ_1 and $-\mu_1$. Consider the map

$$R_x \colon U \oplus V_+ \oplus V_- \to \mathbb{C}^{k+j} \oplus \mathbb{C}^{k+j}$$
$$(f, g_+, g_-) \mapsto (f_-(x) + g_-(x) - f_+(x) - g_+(x), f'_-(x) + g'_-(x) - f'_+(x) - g'_+(x)).$$

This map must be injective (thus bijective) because of the convexity property of solutions given by Equation (42). If $R_x^{-1}(0,u) = (f,g_+,g_-)$, the desired Green's function v can be chosen by taking f outside of $(-\mu_1,\mu_1)$, $f+g_-$ on $(-\mu_1,x)$, and $f+g_+$ on (x,μ_1) .

Lemma 5.5 Let $u_+, u_- \in \mathbb{C}^k$. There is a unique solution to $D_1^*D_1(v) = 0$ on $(-\mu_1, \mu_1)$ with values u_+, u_- at $\mu_1, -\mu_1$ respectively.

Proof: Let W_{-} and W_{+} be the k+j-dimensional affine spaces of solutions with boundary value u_{-} at $-\mu_{1}$, and u_{+} at μ_{1} respectively. Consider at an intermediary point x the affine map

$$R_x \colon W_+ \oplus W_- \to \mathbb{C}^{k+j} \oplus \mathbb{C}^{k+j}$$

 $(g_+, g_-) \mapsto (g_-(x) - g_+(x), g'_-(x) - g'_+(x)).$

It is injective, otherwise one has, taking differences, an element of $\ker(D_1^*D_1)$ vanishing at both ends; it is thus surjective, and the inverse image of zero gives solutions whose values and derivatives match at x. \square

A similar result holds on the other interval.

Lemma 5.6 Let $x \in (\mu_1, \mu_0 - \mu_1)$, $u_+, u_- \in \mathbb{C}^k$ and $u \in \mathbb{C}^k$. There is a unique solution to $D_1^*D_1(v) = \delta_x u$ on the interval, with value u_+ at μ_1 , and value u_- at $\mu_0 - \mu_1$.

Combining the two previous lemmas, we obtain the Green's function for the small side.

Lemma 5.7 Let $x \in (\mu_1, \mu_0 - \mu_1)$ and $u \in \mathbb{C}^k$. There is a unique solution to $D_1^*D_1(v) = \delta_x u$ on the circle.

Proof: Again, one has, varying the boundary values at $\pm \mu_1$, an affine 2k dimensional space of continuous solutions, with the right jump in derivatives at x, and possibly extra jumps in the derivatives at $\pm \mu_1$. One considers the affine map from this space to $\mathbb{C}^k \oplus \mathbb{C}^k$ taking the jumps in the derivatives at $\pm \mu_1$; it has to be injective (convexity) and so is surjective, allowing us to match the derivatives.

By the usual ellipticity argument, the Green's function solution to $D_1^*D_1(v) = \delta_x u$ is smooth away from x. By convexity, ||v|| and $d_z||v||$ are increasing everywhere away from x. Since we are on a circle, both must attain their maximum at x. In addition, integrating over the circle, we get

$$|u|^{2} \geq d_{z} < v, v > (x)_{-} - d_{z} < v, v > (x)_{+}$$

$$= \int d_{z}^{2} < v, v >$$

$$= \int 2\|(\nabla_{0} - i\mu_{0}x_{0})v\|^{2} + 2\sum_{j}\|(T_{j} - ix_{j})v\|^{2}$$

$$\geq C \int \|(\nabla_{0} - i\mu_{0}x_{0})v\|^{2} + \|v\|^{2}.$$

The last inequality follows from the fact that the solution to Nahm's equation is irreducible. This L_1^2 bound on v ensures continuity, with $||v||_{L^{\infty}} \leq C||v||_{L_1^2}$, and hence the L^{∞} norm of the Green's function is bounded by a constant times the norm of u.

The case j=0. We can again use a cohomological version of the Dirac operator, but we must modify the function spaces a little bit to account for the jump at μ_1 in the solution to Nahm's equations given by $\Delta_+(A(\zeta)) = (\alpha_{+,0} + \alpha_{+,1}\zeta)(\bar{\alpha}_{+,1}^T - \bar{\alpha}_{+,0}^T\zeta)$ for suitable column vectors $\alpha_{+,i}$, and the similar jump $\Delta_-(A(\zeta)) = (\alpha_{-,0} + \alpha_{-,1}\zeta)(\bar{\alpha}_{-,1}^T - \bar{\alpha}_{-,0}^T\zeta)$ at $-\mu_1$. The corresponding jumps for the matrices T_i are

$$\Delta_{\pm}(T_3) = \frac{i}{2} (\alpha_{\pm,1} \bar{\alpha}_{\pm,1}^T - \alpha_{\pm,0} \bar{\alpha}_{\pm,0}^T),$$

$$\Delta_{\pm}(T_1 + iT_2) = \alpha_{\pm,0} \bar{\alpha}_{\pm,1}^T.$$

We want solutions to the Dirac equations $D_1^*(s_1, s_2) = D_2(s_1, s_2) = 0$ with jumps that are multiples of $v_{\pm} := (i\alpha_{\pm,1}, \alpha_{\pm,0})$ at $\pm \mu_1$, so we modify the function spaces for Complex (40):

- \tilde{L}_2^2 is the space of sections s of K that are L_2^2 over each interval, continuous at the jumping points, and with a jump discontinuity $\Delta_{\pm}(d_z s) = \frac{1}{2}(\alpha_{\pm,1}\bar{\alpha}_{\pm,1}^T + \alpha_{\pm,0}\bar{\alpha}_{\pm,0}^T)s(\pm\mu_1)$ in derivatives at $\pm\mu_1$;
- $(\tilde{L}_1^2)^{\oplus 2}$ is the space of sections (s_1, s_2) of $K \oplus K$ that are L_1^2 over the intervals, but with a jump discontinuity $\Delta_{\pm}(s_1, s_2) = c_{\pm}v_{\pm}$ at $\pm \mu_1$;
- \tilde{L}^2 is the space of L^2 sections of K.

Proposition 5.8 For solutions to Nahm's equations and for the complex defined using those function spaces just defined, Proposition 5.3 holds, and furthermore, at the jump points,

$$\Delta(d_z(\langle v, v \rangle)) = \langle \Delta(d_z v), v \rangle + \langle v, \Delta(d_z v) \rangle = (\langle \bar{\alpha}_{\pm,0}^T v, \bar{\alpha}_{\pm,0}^T v \rangle + \langle \bar{\alpha}_{\pm,1}^T v, \bar{\alpha}_{\pm,1}^T v \rangle) \ge 0$$
 (43)

for $v \neq 0$ satisfying $D_1^*D_1(v) = 0$; hence, if v is a regular solution vanishing at two points, then v = 0.

Proceeding as for j > 0, one can then build a Green's function. Using the Green function in both cases j > 0 and j = 0, we have the analog of Proposition 5.1.

Proposition 5.9 The cohomology $\ker(D_2)/\operatorname{Im}(D_1)$ of Complex (40) is isomorphic to the space of L^2 solutions to the Dirac equation $D_2(v) = 0 = D_1^*(v)$, which are the harmonic representatives in the cohomology classes. As such, they have minimal norm in the class.

Proof: The cohomology class of a solution to the Dirac equation is non zero as the convexity property ensures $D_1^*D_1(v) = 0$ over the whole circle implies v = 0. On the other hand, from a solution to $D_2(u) = 0$, one uses the Green's function to solve $D_1^*D_1(v) = D_1^*(u)$ and gets a harmonic representative $u + D_1(v)$. \square

We can now turn to identifying the spectral data of a solution to Nahm's equations with the data of the caloron it induces, beginning with the spectral curves. Suppose we are in the generic situation of spectral curves that are reduced and with no common components. Let's work over $\zeta = 0$, and suppose, that the intersection of $\zeta = 0$ with the curves is generic, so distinct points of multiplicity one.

We turn first to analysing the kernel of the matrices $\beta(z) - iw = T_1(z) + iT_2(z) - ix_1 + x_2$, at points w where $\det(\beta(z) - iw) = 0$ so that we are on the Nahm spectral curve, let us say for the interval $(-\mu_1, \mu_1)$. We note that since $[\beta, d_z + \alpha] = 0$, the spectrum is constant along the intervals $(-\mu_1, \mu_1)$ or $(\mu_1, \mu_0 - \mu_1)$, and indeed, if $(\beta(z_0) - iw)f_0 = 0$, solving $(d_z + \alpha)f = 0$, $f(z_0) = f_0$ gives $(\beta(z) - iw)f = 0$ over the interval. For such an f and a bump function ρ , setting

$$(s_1, s_2) = (\rho f, 0) \tag{44}$$

defines a cohomology class in Complex (40). This class is zero only when the integral of ρ is zero over the interval as then for some σ , we have $(d_z + \alpha)(\sigma f) = ((d_z \sigma)f, 0) = (\rho f, 0)$. Note that we can use a boundary and do $\rho \mapsto \rho + d_z \sigma$ to place the bump anywhere on the interval. This fact is quite useful.

Given this class at a fixed y = 0, one can extend it to other y by taking

$$(s_1, s_2) = (e^{iyz} \rho f, 0). \tag{45}$$

We can think of this family of cohomology classes as a section over $S^1 \times \mathbb{R}^3$ of the bundle that Nahm's construction produces from the solution to Nahm's equations. This section has exponential decay as $x_3 \to \pm \infty$. To see this, we exploit the fact that we can move the bump function ρ around to give different representatives supported on $(-\mu_1, \mu_1)$ for the class $(s_1, s_2) = (e^{iyz}\rho f, 0) = (e^{-x_3z}e^{i\mu_0x_0}\rho f)$. Recall that the norm of any representative in the cohomology class is at least the norm of the harmonic (Dirac) representative. As we take x_3 to $+\infty$, we move the bump towards μ_1 , giving up to a polynomial the bound $\exp(-x_3\mu_1)$ on the L^2 norm of the harmonic representative; similarly, as we go to $-\infty$, we move the bump towards $-\mu_1$, and get a bound of the form $\exp(x_3\mu_1)$.

The condition on a section of E over $\mathbb{C}^* = \{e^{i\mu_0x_0+x_3}\}$ to extend over $0, \infty$ is given in [13, Sec. 3], and amounts to a growth condition. More generally, there are a whole family of growth conditions, giving us

not only our bundle E but a family of associated sheaves $E_{(p,q,0)(p',q',\infty)}$, the sheaf of sections of E over \mathcal{T} with poles of order p at \mathcal{T}_0 , with leading term in E_q^0 , and poles of order p' at \mathcal{T}_∞ , with leading term in $E_{q'}^\infty$. The growth condition that $E_{(0,1,0)(0,1,\infty)}$ corresponds to is that of growth bounded (up to polynomial factors) by $\exp(-x_3\mu_1)$ as $x_3 \to \infty$, and by $\exp(x_3\mu_1)$ as $x_3 \to -\infty$; for $E_{(0,0,0)(1,0,\infty)}$, it is growth bounded (up to polynomial factors) by $\exp(x_3\mu_1)$ as $x_3 \to \infty$, and by $\exp(x_3(\mu_0 - \mu_1))$ as $x_3 \to -\infty$. Let us consider the case of $E_{(0,1,0)(0,1,\infty)}$. Our bounds for the section (45) tell us that it is a (holomorphic) has a global section of $E_{(0,1,0)(0,1,\infty)}$ on the \mathbb{P}^1 above $(\eta,\zeta)=(w,0)$.

The sheaves $E_{(p,q,0)(p',q',\infty)}$ have degree 2(p+p'-2)+q+q' on the fibers of the projections to $T\mathbb{P}1$, from our identifications of them given above. In particular, the degree of $E_{(0,1,0)(0,1,\infty)}$ is -2. Now, starting from a point of the Nahm spectral curve, we have produced a global section of $E_{(0,1,0)(0,1,\infty)}$ over the corresponding Riemann sphere; the Riemann–Roch theorem then tells us that the first cohomology is also non-zero, so that we are in the support of $R^1\tilde{\pi}_*(E_{(0,1,0)(0,1,\infty)})$, which is exactly the caloron spectral curve S_1 . As both S_1 are of the same degree and unreduced, they are identical.

Proceeding similarly for the spectral Nahm spectral curve S_0 , we obtain a global section of $E_{(0,0,0)(1,0,\infty)}$ over the corresponding twistor line, and so show that the line lies in the support of $R^1\tilde{\pi}_*(E_{(0,0,0)(1,0,\infty)})$, which is exactly the caloron spectral curve S_0 .

We have now identified the spectral curves for the solutions to Nahm's equations and the caloron that it produces. We now note that, in addition to the bundle V over $S^1 \times \mathbb{R}^3$ that Nahm's construction produces, there is another bundle, \hat{V} , obtained by taking the spectral data associated to the solution of Nahm's equations, and feeding it in to the reconstruction of a caloron from its spectral data. Indeed, from a solution to Nahm's equations, we have seen that we can can define sheaves \mathcal{L}_z ; the fact that they satisfy the boundary conditions by Proposition 4.2 tells us we can reconstruct a bundle F of infinite rank over $T\mathbb{P}^1$ using Sequence (10). We now identify the bundles V and \hat{V} .

Starting from a caloron, we obtained F as the pushdown from the twistor space \mathcal{T} of a rank 2 bundle E. If w is a fibre coordinate on $\mathcal{T} \to T\mathbb{P}^1$, we saw that it induced an automorphism W of F, such that, at $w = w_0$, $E_{w_0} \simeq F/\text{Im}(W - w_0)$. In Sequence (10), the shift map W identifies the nth entry in the middle column with the n+2th (e.g. $L^{(p-1)\mu_0+\mu_1}(2k+j)\otimes\mathcal{I}_{S_{01}}$ with $L^{p\mu_0-\mu_1}(2k+j)\otimes\mathcal{I}_{S_{10}}$) and similarly in the right hand column. If $w = w(\zeta)$ is the equation of a twistor line L in \mathcal{T} , then quotienting by $W - w(\zeta)$ in Sequence (10) gives for E over the line (recalling that L_0^μ is trivial over the line)

$$E \hookrightarrow \begin{array}{cccc} L^{\mu_1}(2k+j) \otimes \mathcal{I}_{S_{01}} & L^{\mu_1}(2k+j)[-S_{01}]|_{S_0} \\ \oplus & \oplus & \oplus & \to & 0. \\ L^{-\mu_1}(2k+j) \otimes \mathcal{I}_{S_{10}} & L^{-\mu_1}(2k+j)[-S_{10}]|_{S_1} \end{array}$$
(46)

From the twistor point of view, the space of global sections of E along the real line L_x in \mathcal{T} correspond to the fiber of the caloron bundle \hat{V} over the corresponding $x \in S^1 \times \mathbb{R}^3$. On the other hand, from the Nahm point of view, V_x correspond to ker D_x^* , and so to the cohomology $\ker(D_2)/\operatorname{Im}(D_1)$ in Complex (40).

To identify V and \hat{V} , we follow closely [18, pp. 80–84], so we simply summarise the ideas. The identification is made for the x whose lines L_x which do not intersect $S_0 \cap S_1$; to do it on this dense set is sufficient, as one is also identifying the connections.

Let us denote the sequence of sections corresponding to (46) by

$$0 \to H^0(L_x, E) \to U_{\mu_1} \oplus U_{-\mu_1} \to W_0 \oplus W_1. \tag{47}$$

What [18] does is to identify W_0 and W_1 with solutions to $D_x^*(s) = 0$ on the interval $(\mu_1, \mu_0 - \mu_1)$, and $(-\mu_1, \mu_1)$ respectively, and U_{μ_1} and $U_{-\mu_1}$ with L^2 -solutions on a neighborhood of μ_1 and $-\mu_1$ respectively. The kernel $H^0(L_x, E)$ then gets identified with global L^2 solutions on the circle, which are elements of V_x . Since from the twistor point of view $\hat{V}_x \simeq H^0(L_x, E)$, we are done. The case j = 0 is similar.

Having identified the bundles, one then wants to ensure that the connections defined in both case are the same, and we do so again according to [18]. It suffices to do this identification along one null plane through the point x, as changing coordinates will do the rest, and we can suppose that x is the origin, so $\eta = 0$ in $T\mathbb{P}^1$. Let us choose the plane corresponding to $\zeta = 0$. From the twistor point of view, parallel sections along this null plane correspond to sections on the corresponding family of lines with fixed values at ζ

= 0. From the Nahm point of view, a parallel section along this null plane is a family of cohomology classes represented by cocycles $s=(s_1,s_2)$ in $\ker(D_2)$ that satisfy $(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2})s=0, (\frac{\partial}{\partial x_0}-i\frac{\partial}{\partial x_3})s=0$, modulo coboundaries. In particular, the derivatives of s_2 are of the form $(T_1+iT_2)\phi=A_0\phi$ for a suitable ϕ depending on x. Following through the identifications of [18, pp. 80–84] tells us that the derivatives $\nabla \hat{s}$ in suitable trivialisations of the corresponding sections \hat{s} on the family of lines are of the form $\nabla \hat{s} = A_0 \psi$ for the corresponding section ψ . However, the defining relation for $A(\zeta)$ is $(\eta \mathbf{I} - A_0 - \zeta A_1 - \zeta^2 A_2)\phi = 0$. In particular, that at $(\zeta, \eta) = 0$, $A_0 \phi$ vanishes as a function, and so \hat{s} is parallel, as desired.

5.3 From Nahm to caloron to Nahm to caloron.

Let us first place ourselves in the generic situation. We have seen first that the family of generic calorons maps continuously and injectively into the family of generic spectral data; secondly, that generic spectral data and generic solutions to Nahm's equations are equivalent; thirdly, that starting from a caloron and producing a solution to Nahm's equations then taking the spectral data of both objects gives the same result, and fourthly that the generic solution to Nahm's equation gives the same caloron, whether you pass through the Nahm transform or through the spectral data via the twistor construction. Taken together, those facts tell us that the map from caloron to spectral data is a bijection, that all three sets of data are equivalent, and that the six transforms between them are pairwise inverses of each other.

Let us now leave the generic set. If we now take an arbitrary solution T to Nahm's equations, again satisfying all the conditions, we can fit it into a continuous family T(t) with T(0) = T with T(t) generic for $t \neq 0$; that it can be done follows from our description of moduli given in Section 6. The Nahm transform of this family is a continuous family C(t) of calorons, which in turn produces a continuous family of solutions $\tilde{T}(t)$ to Nahm's equations. We do not know a priori whether the boundary and symmetry conditions are satisfied at t = 0. However, for $t \neq 0$, $\tilde{T}(t) = T(t)$, and so $\tilde{T}(0) = T(0)$, and we are done; the transform Nahm to caloron to Nahm is the identity.

If the caloron C lies in the closure of the set of generic calorons (presumably all calorons do, but it needs to be proved, perhaps using the methods of Taubes ([31]) to show that the moduli space is connected), we again fit it as C(0) into a family C(t) of calorons, with C(t) generic for $t \neq 0$, all of same charge. The Nahm transform produces a family T(t) of solutions to Nahm's equations, with T(t) satisfying all the conditions for $t \neq 0$. We also get a family S(t) of spectral data, which for all t corresponds to both C(t) and T(t). We need to show that T(0) satisfy all the conditions.

Taking a limit, it is fairly clear that the symmetry condition is satisfied also for t=0. The boundary conditions are less obvious. From the small side, there is no problem, as the vanishing theorem 3.1 holds at C(0), and so the solutions to Nahm's equations are continuous at the boundary of the interval. From the large side, what saves us is the rigidity of representations into SU(2); indeed, the polar parts of the solutions are given by representations, and so are fixed, in a suitable family of unitary gauges, hence preserved in a limit. The process, given in Section 4.2, of passing to a constant gauge by solving $ds/dz + A_{+}(z)s = 0$ applies in the limit near the singular points. We have in the limit the same type of transformations relating the basis of K in which one has the solution to Nahm's equations to the continuous basis obtained from lifting up and pushing down as to obtain Sequence (16), with the same process of producing endomorphisms $\underline{A}(z,\zeta)(t)$, giving us again in the limit a continuous endomorphism of a bundle V over $T\mathbb{P}^1\times S^1$, with V in fact lifted from $\mathbb{P}^1 \times S^1$. The restrictions $V(\pm \mu_1)$ (on the large side) are of fixed type $V_{\mu_1}^1 = \mathcal{O}^k \oplus V_{j-1,-j+1}$ for all t. The summand $V_{j-1,-j+1}$ corresponds to the subsheaf $\mathcal{O}(j)$ of the sheaf $F/(F_{p,1}^0 + F_{-p,1}^\infty) \otimes L^{\pm \mu_1}$, which exists in the limit. The polar part $\mathcal{C}(p(\zeta,t))$ of Equation (26), mapping $V_{j-1,-j+1}$ to itself, has limit $\mathcal{C}(p(\zeta,0))$. By Equation (27), this latter limit is determined by the spectral curves, and so is well defined. The other summand \mathcal{O}^k is also well defined in the limit, as it is the piece transferred from the small side, which is still \mathcal{O}^k in the limit because of the vanishing theorem. The off-diagonal vanishing as one goes back to the trivialisation for Nahm's equations is simply a consequence of the polar behaviour of solutions to $ds/dz + A_{+}(z)s = 0$, and of the normal form for $A(z,\zeta)(t)$ as in Equation (25). In short, the limit has exactly the same normal form, and so the same boundary behaviour.

Finally, there is the question of irreducibility. The reducible solutions correspond to calorons for which charge has bubbled off; as our family has constant charge, this is precluded. The limit solutions satisfies

all the conditions necessary to produce a caloron by the opposite Nahm transform; as the transform is involutive on the generic member of the family, it is also involutive in the limit, and so the circle closes. In short one has the following theorem.

Theorem 5.10 A) There is an equivalence between

- 1. Generic calorons of charge (k, j);
- 2. Generic solutions to Nahm's equations on the circle satisfying the conditions of Section 2.1.B;
- 3. Generic spectral data.

The equivalences of 1. and 2. are given in both directions by the Nahm transform.

- B) The Nahm transforms give equivalences between
 - 1. Calorons of charge (k, j), in the closure of the generic set;
 - 2. Solutions to Nahm's equations on the circle satisfying the conditions of Section 2.1.B.

6 Moduli

The equivalence exhibited above allows us to classify calorons by classifying appropriate solutions to Nahm's equations. One has to guide us the example of monopoles, as classified in [11, 17]; the (framed) monopoles for gauge group G, with symmetry breaking to a torus at infinity, charge k are classified by the space $Rat_k(\mathbb{P}^1, G_{\mathbb{C}}/B)$ of based degree k rational maps from the Riemann sphere into the flag manifold $G_{\mathbb{C}}/B$. As our calorons are Kač-Moody monopoles, we should have the same theorem, with framed calorons equivalent to rational maps from \mathbb{P}^1 into the loop group. Following an idea developed in [2], one thinks of these as bundles on $\mathbb{P}^1 \times \mathbb{P}^1$, with some extra data of a flag along a line, and some framing.

The rank 2 bundle corresponding to an SU(2) caloron, again following the example of monopoles, should be the restriction of the bundle E on $\tilde{\mathcal{T}}$ corresponding to the caloron, to the inverse image of a point, say $\zeta = 0$, in \mathbb{P}^1 . This inverse image is $\mathbb{P}^1 \times \mathbb{C}$; along the divisor $\{0\} \times \mathbb{C}$, the bundle has the flag E_0 , and along $\{\infty\} \times \mathbb{C}$, the bundle has the flag E_{∞} . We extend this bundle to infinity, using a framing, in such a way that E_{∞} is a trivial subbundle, and E_0 has degree -j. This bundle will be corresponding to the caloron.

The theorem, however, is proven in terms of solutions to Nahm's equations; indeed, we show in [8] that both bundles and solutions to Nahm's equations satisfying the appropriate conditions are describable in terms of a geometric quotient of a family of matrices.

Theorem 6.1 Let $k \geq 1, j \geq 0$ be integers. There is an equivalence between

- 1) vector bundles E of rank two on $\mathbb{P}^1 \times \mathbb{P}^1$, with $c_1(E) = 0$, $c_2(E) = k$, trivialised along $\mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1$, with a flag $\phi \colon \mathcal{O}(-j) \hookrightarrow E$ of degree j along $\mathbb{P}^1 \times \{0\}$ such that $\phi(\infty)(\mathcal{O}(-j)) = \operatorname{span}(0,1)$;
- 2) framed, irreducible solutions to Nahm's equations on the circle, with rank k over $(\mu_1, \mu_0 \mu_1)$, rank k + j over $(-\mu_1, \mu_1)$, with the boundary conditions defined above, modulo the action of the unitary gauge group;
- 3) complex matrices A, B $(k \times k), C$ $(k \times 2), D$ $(2 \times k), A'$ $(j \times k), B'$ $(1 \times k), C'$ $(j \times 2),$ satisfying the monad equations

$$[A, B] + CD = 0,$$

$$\binom{B'}{0}A + S(j, 0)A' - A'B - C'D = 0,$$

$$-e_{+}A' + \begin{pmatrix} 1 & 0 \end{pmatrix}D = 0,$$

and the genericity conditions

$$\begin{pmatrix} A-y \\ B-x \\ D \end{pmatrix} \text{ is injective for all } x,y \in \mathbb{C},$$

$$(x-B \quad A-y \quad C) \text{ is surjective for all } x,y \in \mathbb{C},$$

$$\begin{pmatrix} x-B \quad A \quad C & 0 \\ \begin{pmatrix} -B' \\ 0 \end{pmatrix} \quad A' \quad C' \quad x-S(j,0) \\ 0 \quad 0 \quad (1 \quad 0) \quad -e_+ \end{pmatrix} \text{ is surjective for all } x \in \mathbb{C},$$

$$\begin{pmatrix} \begin{pmatrix} A \\ A' \end{pmatrix} \quad \begin{pmatrix} C_2 \\ C_2' \end{pmatrix} \quad M \begin{pmatrix} C_2 \\ C_2' \end{pmatrix} \quad \cdots \quad M^{j-1} \begin{pmatrix} C_2 \\ C_2' \end{pmatrix} \end{pmatrix} \text{ is an isomorphism,}$$

modulo the action of $Gl(k,\mathbb{C})$ given by

$$(A,B,C,D,A',B',C') \mapsto (gAg^{-1},gBg^{-1},gC,Dg^{-1},A'g^{-1},B'g^{-1},C').$$

Here C_i , C'_i are the ith column of C, C', and D_i the ith row of D, Equation (32) gives S(j,0), and $e_+ := \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$,

$$M := \begin{pmatrix} B & -C_1 e_+ \\ \begin{pmatrix} B' \\ 0 \end{pmatrix} & S(j,0) - C'_1 e_+ \end{pmatrix}.$$

We see in [8] that B and M are conjugate to the matrix $T_1 + iT_2$ on the intervals $(\mu_1, \mu_0 - \mu_1)$ and $(-\mu_1, \mu_1)$ respectively. Choosing B diagonal, and B and M with distinct and disjoint eigenvalues, it is not difficult to build solutions to the various matrix constraints. Hence the spectral curves for the obtained solutions to Nahm's equations are reduced and have no common components, as the spectral curve over $\zeta = 0$ is the spectrum of $T_1 + iT_2$. Thus generic generic solutions to Nahm's equations, and so generic calorons, exist.

References

- [1] M. R. Adams, J. Harnad, and J. Hurtubise. Isospectral Hamiltonian flows in finite and infinite dimensions. II. Integration of flows. *Comm. Math. Phys.*, 134(3):555–585, 1990.
- [2] M. F. Atiyah. Instantons in two and four dimensions. Comm. Math. Phys., 93(4):437–451, 1984.
- [3] M. F. Atiyah, V. G. Drinfel'd, N. J. Hitchin, and Y. I. Manin. Construction of instantons. *Phys. Lett.* A, 65(3):185–187, 1978.
- [4] C. P. Boyer, B. M. Mann, J. C. Hurtubise, and R. J. Milgram. The topology of the space of rational maps into generalized flag manifolds. *Acta Math.*, 173(1):61–101, 1994.
- [5] P. J. Braam and P. van Baal. Nahm's transformation for instantons. Comm. Math. Phys., 122(2):267–280, 1989.
- [6] B. Charbonneau. Analytic aspects of periodic instantons. PhD thesis, Massachusetts Institute of Technology, 2004. (see www.math.mcgill.ca/~charbonneau).
- [7] B. Charbonneau. From spatially periodic instantons to singular monopoles. Comm. Anal. Geom., 14(1):183–214, 2006, arXiv:math.DG/0410561.
- [8] B. Charbonneau and J. Hurtubise. Calorons, Nahm's equations on S^1 and bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, 2006, arXiv:math.DG/0610804.
- [9] S. Cherkis and A. Kapustin. Nahm transform for periodic monopoles and $\mathcal{N}=2$ super Yang–Mills theory. Comm. Math. Phys., 218(2):333–371, 2001, arXiv:hep-th/0006050v2.
- [10] E. Corrigan and P. Goddard. Construction of instanton and monopole solutions and reciprocity. *Ann. Physics*, 154(1):253–279, 1984.

- [11] S. K. Donaldson. Nahm's equations and the classification of monopoles. Comm. Math. Phys., 96(3):387–407, 1984.
- [12] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990.
- [13] H. Garland and M. K. Murray. Kac–Moody monopoles and periodic instantons. *Comm. Math. Phys.*, 120(2):335–351, 1988.
- [14] J. Gravesen. On the topology of spaces of holomorphic maps. Acta Math., 162(3-4):247–286, 1989.
- [15] N. J. Hitchin. Monopoles and geodesics. Comm. Math. Phys., 83(4):579-602, 1982.
- [16] N. J. Hitchin. On the construction of monopoles. Comm. Math. Phys., 89(2):145–190, 1983.
- [17] J. Hurtubise. The classification of monopoles for the classical groups. Comm. Math. Phys., 120(4):613–641, 1989.
- [18] J. Hurtubise and M. K. Murray. On the construction of monopoles for the classical groups. *Comm. Math. Phys.*, 122(1):35–89, 1989.
- [19] J. Hurtubise and M. K. Murray. Monopoles and their spectral data. Comm. Math. Phys., 133(3):487–508, 1990.
- [20] M. Jardim. Construction of doubly-periodic instantons. Comm. Math. Phys., 216(1):1–15, 2001, arXiv:math.DG/9909069.
- [21] M. Jardim. Classification and existence of doubly-periodic instantons. Q. J. Math., 53(4):431–442, 2002, arXiv:math.DG/0108004.
- [22] M. Jardim. Nahm transform and spectral curves for doubly-periodic instantons. Comm. Math. Phys., 225(3):639–668, 2002, arXiv:math.AG/9909146.
- [23] M. Jardim. A survey on Nahm transform. J. Geom. Phys., 52(3):313-327, 2004, arXiv:math.DG/0309305.
- [24] R. Mazzeo and R. B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian J. Math.*, 2(4):833–866, 1998, arXiv:math.DG/9812120.
- [25] W. Nahm. All self-dual multimonopoles for arbitrary gauge groups. In Structural elements in particle physics and statistical mechanics (Freiburg, 1981), volume 82 of NATO Adv. Study Inst. Ser. B: Physics, pages 301–310. Plenum, New York, 1983.
- [26] W. Nahm. Self-dual monopoles and calorons. In *Group theoretical methods in physics (Trieste, 1983)*, volume 201 of *Lecture Notes in Phys.*, pages 189–200. Springer, Berlin, 1984.
- [27] T. M. W. Nye. The geometry of calorons. PhD thesis, U. of Edinburgh, 2001. arxiv:hep-th/0311215.
- [28] T. M. W. Nye and M. A. Singer. An L^2 -index theorem for Dirac operators on $S^1 \times \mathbf{R}^3$. J. Funct. Anal., 177(1):203–218, 2000, arXiv:math.DG/0009144.
- [29] H. Schenk. On a generalised Fourier transform of instantons over flat tori. Comm. Math. Phys., 116(2):177–183, 1988.
- [30] S. Szabo. Nahm transform for integrable connections on the Riemann sphere. PhD thesis, IRMA, Université de Strasbourg, 2005, arXiv:math.DG/0511471.
- [31] C. H. Taubes. Path-connected Yang-Mills moduli spaces. J. Differential Geom., 19(2):337–392, 1984.

Department of Mathematics and Statistics, McGill University 805 Sherbrooke St. W, Montreal, Quebec, H3A 2K6, Canada. E-mails: benoit@alum.mit.edu and jacques.hurtubise@mcgill.ca